Pointwise Min-Norm Control

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Objective: In this lecture, we aim to design a controller to provide exponentially convergence of the system states to zero while the control signal is minimized.

Consider the following affine linear system:

$$\dot{x} = Ax + Bu \tag{1}$$

where the matrix pair (A, B) is controllable.

Choose symmetric positive definite matrices Q and R, and let P be the symmetric positive definite solution to the continuous algebraic Riccati equations (CARE)

$$A^T P + PA - PBR^{-1}B^T P + Q = 0 \tag{2}$$

Define the following positive-definite Lyapunov function

$$V(x) = x^T Px \tag{3}$$

Taking derivative of Eq. (3) and substituting Eq. (1) into it yields

$$\dot{V}(x) = 2x^T P\dot{x} = 2x^T PAx + 2x^T PBu = x^T(A^T P + PA)x + (2x^T PB)u \tag{4}$$

Then, to achieve the exponentially convergence of the solution, the following condition is desired

$$\dot{V}(x) \leq -\epsilon V(x) \tag{5}$$

where $\epsilon > 0$ determines the convergence rate. Tuning of ϵ can make a tradeoff between the control effort and the rate of convergence of the states to zero.

Substituting Eq. (4) into Eq. (5) gives the following condition

$$x^T(A^T P + PA)x + (2x^T PB)u + \epsilon V(x) \leq 0 \tag{6}$$

The above condition is called control Lyapunov function (CLF) constraint. Now, the minimal selection of an admissible control u is desired to be designed subject to the CLF constraint of Eq. (6). Such admissible control is called pointwise min-norm control (PWMC) and should be defined as

$$u = \{\text{argmin}\|u\| \ s.t. \text{ CLF constraint satisfied}\} \tag{7}$$

The following PWMC law is a solution to the minimization problem

$$u = \begin{cases} -\frac{\phi_0(x)\phi_1(x)}{\phi_1^T(x)\phi_1(x)} & \text{if } \phi_0(x) > 0 \\ 0 & \text{if } \phi_0(x) \leq 0 \end{cases}, \tag{8}$$

where ϕ_0 and ϕ_1 are defined as

$$\begin{aligned} \phi_0(x) &= x^T(A^TP + PA)x + \epsilon V(x) \\ \phi_1(x) &= 2x^TPB \end{aligned} \tag{9}$$

Remark 1: Note that the Lyapunov function defined in Eq. (3) satisfies the following condition:

$$\begin{aligned} \lambda_{min}(P)\|x\|^2 \leq V(x) \leq \lambda_{max}(P)\|x\|^2 \\ \lambda_{min}(P)\|x(0)\|^2 \leq V(x(0)) \leq \lambda_{max}(P)\|x(0)\|^2 \end{aligned} \tag{10}$$

and solution of Eq. (5) is

$$V(x(t)) \leq e^{-\epsilon t}V(x(0)) \tag{11}$$

Combining Eq. (11) with Eq. (10) gives

$$\|x\| \leq \beta e^{-\frac{\epsilon}{2}t}\|x(0)\| \tag{12}$$

where $\beta = \sqrt{\frac{\lambda_{max}(P)}{\lambda_{min}(P)}}$. Eq. (12) shows that using the designed PWMC that satisfies the inequality of Eq. (11), the system states exponentially converge to zero with the converge rate $\frac{\epsilon}{2}$.

Remark 2: Consider the general affine nonlinear systems,

$$\dot{x} = f(x) + Bu \tag{13}$$

To use the PWMC law for the nonlinear system of Eq. (13), feedback linearization should be first applied to the above system to achieve an affine linear system in the form of Eq. (1) (if system of Eq. (13) is a feedback linearizable system).

Remark 3: Further details on PWMC and CLF can be found in a book entitled Robust Nonlinear Control Design by Randy A. Freeman et al., 1996.