EXAMPLES OF CONTINUITY USING ϵ-δ DEFINITION.

$\boxed{1}$ $f(x) = x^2$ continuity at $x_0 = 0$.

$$|x - x_0| \leq \delta \Rightarrow |x^2 - x_0^2| \leq \epsilon$$

$$|x| \leq \delta \Rightarrow |x^2| \leq \epsilon$$

$$\Updownarrow$$

$$|x|^2 \leq \epsilon$$

$$\Updownarrow$$

$$|x| \leq \epsilon^{1/2}$$

$$\Updownarrow$$

pick $\delta = \epsilon^{1/2}$ so $\delta(\epsilon; 0) = \epsilon^{1/2}$

$\boxed{2}$ $f(x) = \sqrt{x}$ continuity at $x_0 > 0$.

domain $D = \{x \mid x > 0\}$

$$|\sqrt{x} - \sqrt{x_0}| < \epsilon$$

NOTE: $x - x_0 = (\sqrt{x} - \sqrt{x_0})(\sqrt{x} + \sqrt{x_0}) \Rightarrow \sqrt{x} - \sqrt{x_0} = \dfrac{x - x_0}{\sqrt{x} + \sqrt{x_0}}$

$\Leftrightarrow$

$$|\sqrt{x} - \sqrt{x_0}| = \left| \frac{x - x_0}{\sqrt{x} + \sqrt{x_0}} \right| < \epsilon$$

$\Leftarrow$

NOW $\left| \dfrac{x - x_0}{\sqrt{x} + \sqrt{x_0}} \right| < \left| \dfrac{x - x_0}{\sqrt{x_0}} \right|$

at this step the process now goes only one way, hence the implication in blue as opposed to the equivalence (they are in black)

LET

$$\left| \frac{x - x_0}{\sqrt{x_0}} \right| < \epsilon$$

$\Leftrightarrow$

$|x - x_0| < \epsilon\sqrt{x_0}$ pick $\delta(\epsilon; x_0) = \epsilon\sqrt{x_0}$

3] $f(x) = e^{x_0}$ continuity at $x_0 \in \mathbb{R}$.

$$|e^x - e^{x_0}| \le \epsilon$$

$$\Leftrightarrow$$

$$|e^{x_0}(e^{x-x_0} - 1)| \le \epsilon$$

$$\Leftrightarrow \quad e^{x_0} > 0$$

$$|e^{x-x_0} - 1| \le \epsilon / e^{x_0}$$

$$\Leftrightarrow$$

MUST SATISFY
$$\left\{\begin{array}{lll} \text{A. if} & x > x_0 & |e^{|x-x_0|} - 1| \le \epsilon/e^{x_0} \\ \text{B. if} & x < x_0 & |e^{-|x-x_0|} - 1| \le \epsilon/e^{x_0} \end{array}\right.$$

A. $|e^{|x-x_0|} - 1| \le \epsilon/e^{x_0}$

$$\Leftrightarrow \quad e^{|x-x_0|} - 1 \le \epsilon/e^{x_0}$$

$$\Leftrightarrow \quad e^{|x-x_0|} \le 1 + \epsilon/e^{x_0}$$

$$\Leftrightarrow \quad |x - x_0| \le \ln(1 + \epsilon/e^{x_0})$$

B. $|e^{-|x-x_0|} - 1| \le \epsilon/e^{x_0}$

$$\Leftrightarrow \quad 1 - e^{-|x-x_0|} \le \epsilon/e^{x_0}$$

$$\Leftrightarrow \quad e^{-|x-x_0|} \ge 1 - \epsilon/e^{x_0}$$

$$\Leftrightarrow \quad -|x-x_0| \ge \ln(1 - \epsilon/e^{x_0})$$

$$\Leftrightarrow \quad |x - x_0| \le -\ln(1 - \epsilon/e^{x_0})$$

$$\Leftarrow$$

$$|x - x_0| \le \min\left(\ln(1+\epsilon/e^{x_0}),\ -\ln(1-\epsilon/e^{x_0})\right)$$

$$\Leftrightarrow \qquad \leftarrow \text{since } \ln(1+\epsilon/e^{x_0}) < -\ln(1-\epsilon/e^{x_0}).$$

$$|x - x_0| \le \ln(1 + \epsilon/e^{x_0})$$

pick $\delta(\epsilon; x_0) = \ln(1 + \epsilon/e^{x_0})$

alternative version

$$|e^{x} - e^{x_0}| \le \epsilon$$

$$\Longleftrightarrow$$

$$-\epsilon \le e^{x} - e^{x_0} \le \epsilon$$

$$\Longleftrightarrow$$

$$-\epsilon/e^{x_0} \le e^{x-x_0} - 1 \le \epsilon/e^{x_0}$$

$$\Longleftrightarrow$$

$$1 - \epsilon/e^{x_0} \le e^{x-x_0} \le 1 + \epsilon/e^{x_0}$$

$$\Longleftrightarrow$$

$$\ln(1-\epsilon/e^{x_0}) \le x - x_0 \le \ln(1+\epsilon/e^{x_0})$$

$$\Longleftarrow$$

$$|x - x_0| \le \min\left(\ln(1+\epsilon/e^{x_0}),\ -\ln(1-\epsilon/e^{x_0})\right)$$

$$\Longleftrightarrow$$

$$|x - x_0| \le \ln(1+\epsilon/e^{x_0})$$

pick $\delta(\epsilon; x_0) = \ln(1+\epsilon/e^{x_0})$

4] $f(x) = \ln(x)$ $\qquad x, x_0 > 0$

$D = \{x \mid x > 0\}$

$$|\ln(x) - \ln(x_0)| \le \epsilon$$

$$\Leftrightarrow$$

$$|\ln(x/x_0)| \le \epsilon$$

$$\Leftrightarrow$$

$$-\epsilon \le \ln(x/x_0) \le \epsilon$$

$$\Leftrightarrow$$

$$e^{-\epsilon} \le \frac{x}{x_0} \le e^{\epsilon}$$

$$\Leftrightarrow$$

$$e^{-\epsilon} - 1 \le \frac{x}{x_0} - 1 \le e^{\epsilon} - 1$$

$$\Leftrightarrow$$

$$x_0(e^{-\epsilon} - 1) \le x - x_0 \le x_0(e^{\epsilon} - 1)$$

$$\Leftrightarrow$$

$$-x_0(1 - e^{-\epsilon}) \le x - x_0 \le x_0(e^{\epsilon} - 1)$$

$$\Leftarrow$$

$$|x - x_0| \le x_0(e^{\epsilon} - 1)$$

AND

$$|x - x_0| \le x_0(1 - e^{-\epsilon})$$

$$\Leftarrow$$

$$|x - x_0| \le x_0 \min(e^{\epsilon} - 1, 1 - e^{-\epsilon})$$

smaller of the two.

$$\Leftrightarrow$$

$$|x - x_0| \le x_0(1 - e^{-\epsilon})$$

pick $\delta(\epsilon; x_0) = x_0(1 - e^{-\epsilon})$