# Adaptive Control using Projection Operator.

Let's consider a case where some error is gauranteed and analyze the system response using the projector operator in the adaptive laws. Obviously many other sources of disturbances can benefit from the use of the projection operator.

**Plant:**

$$\dot{x}(t) = A\,x(t) + b\lambda\left[u(t) + f(x(t))\right]$$

suppose $f$ is <u>not</u> linearly parametrizable.

$\Rightarrow$

$$f(x(t)) = \underbrace{\alpha\,\Phi(x(t))}_{\text{parametrized approximation}} + \underbrace{\epsilon(x(t))}_{\text{error}}$$

$\Rightarrow$

$$\dot{x}(t) = A x(t) + b\lambda\left[u(t) + \alpha\Phi(x(t)) + \epsilon(x(t))\right]$$

**Model:**

$$\dot{x}_m(t) = A_m x_m(t) + b_m r(t)$$

**Controller:**

$$u(t) = k_x^T(t)\,x(t) + k_r(t)\,r(t) - \hat{\alpha}(t)\,\Phi(x(t))$$

MATCHING ASSUMPTIONS:

$$\exists\, k_x^* : \; b\lambda (k_x^*)^T = A_m - A$$

$$\exists\, k_r^* : \; b\lambda k_r^* = b_m$$

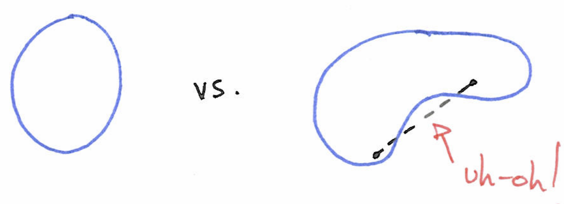
# Projection-Based Adaptation

- provides a method to smoothly adjust adaptation so that it remains within a compact, convex domain.

- What does it mean for a domain to be convex?

domain $\Omega$ is convex if

$$\forall\, \theta_1, \theta_2 \in \Omega \subset \mathbb{R}^n, \qquad \lambda\theta_1 + (1-\lambda)\theta_2 \in \Omega \quad \forall\, 0 \le \lambda \le 1.$$




- What does it mean for a function $\Psi : \mathbb{R}^n \to \mathbb{R}$ to be convex?

$$\forall\, \theta_1, \theta_2 \in \mathbb{R}^n, \qquad f(\lambda\theta_1 + (1-\lambda)\theta_2) \le \lambda f(x) + (1-\lambda) f(y)$$

- Convexity of $\Psi : \mathbb{R}^n \to \mathbb{R}$ means that its level sets are convex.
(+ interior)

Lemma. $\Psi : \mathbb{R}^n \to \mathbb{R}$ convex $\Rightarrow$ $\Omega_\delta = \{\theta \in \mathbb{R}^n \mid \Psi(\theta) \le \delta\}$ is convex for $\delta > 0$.

- Convexity of $\Psi : \mathbb{R}^n \to \mathbb{R}$ means that its gradient is in the direction of the outward normal to the level sets of $\Psi$.
of course $\Psi$ must be differentiable.

**Lemma.** Let $\Psi : \mathbb{R}^n \to \mathbb{R}$ be $C^1(\mathbb{R}^n; \mathbb{R})$. For a constant $\delta > 0$ and its associated $\Omega_\delta$, let $\theta_i, \theta_b \in \Omega_\delta$ be such that $\Psi(\theta_i) < \delta$ (interior point) and $\Psi(\theta_b) = \delta$ (boundary point). Then,

$$\nabla \Psi(\theta_b) \cdot (\theta_b - \theta_i) \leq 0$$

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The Projection operator:

**Definition.** Consider a convex compact set with a smooth boundary given by $\Omega_c = \{\theta \in \mathbb{R}^n \mid \Psi(\theta) \leq c\}$, $0 \leq c \leq 1$, where $\Psi : \mathbb{R}^n \to \mathbb{R}$ is the following smooth convex function

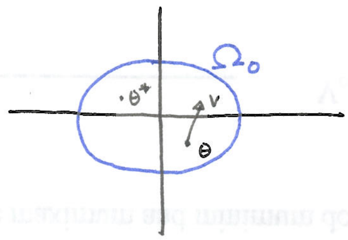
$$\Psi(\theta) \equiv \frac{\theta^T \theta - \theta_{max}^2}{\epsilon_\theta \, \theta_{max}^2}$$

where $\theta_{max}$ is the norm bound imposed on the parameter $\theta$ and $\epsilon_\theta > 0$ denotes the convergence tolerance of choice. Let the true value of the parameter be $\theta^* \in \Omega_0$. The projection operator is defined as

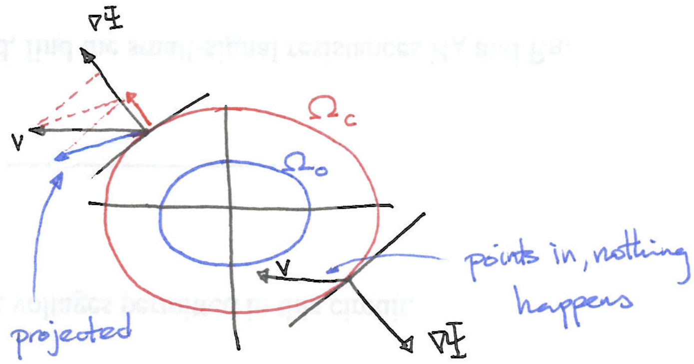
$$\mathrm{Proj}(\theta, v) = \begin{cases} v & \text{if } \Psi(\theta) < 0 \\ v & \text{if } \Psi(\theta) \geq 0 \text{ and } \langle \nabla\Psi, v \rangle \leq 0 \\ v - \Psi(\theta) \left\langle \dfrac{\nabla \Psi}{\|\nabla \Psi\|}, v \right\rangle \dfrac{\nabla \Psi}{\|\nabla \Psi\|} & \text{if } \Psi(\theta) \geq 0 \text{ and } \langle \nabla\Psi, v \rangle > 0 \end{cases}$$

- $\Psi(\theta)$ can be written as $\Psi(\theta) \equiv \dfrac{\langle \theta, \theta \rangle - \theta_{max}^2}{\epsilon_\theta \, \theta_{max}^2}$.

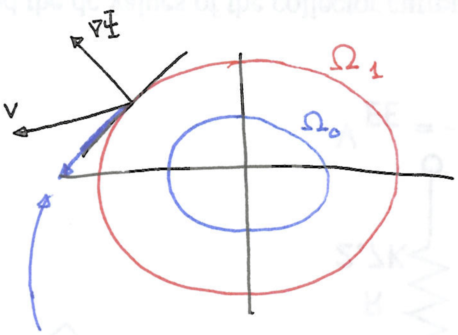
How projection operator works:



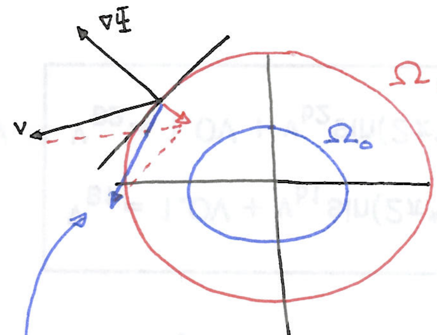

it's OK, nothing happens




$0 < c < 1$




projected to be tangent to level set.




$c > 1$

projected to point towards the interior of $\Omega_c$.

idea is to smoothly modify vector to point inwards (if it doesn't) as we get further from $\Omega_0$. We take drastic measures outside of $\Omega_1$ to get it to point in.

- If $\dot{\theta}(t) = \mathrm{Proj}(\theta(t), v(t))$ for some $v(t)$, then $\theta(t)$ never leaves $\Omega_1$ if $\theta(0) \in \Omega_1$.

Property. Given the parameter state $\theta$ and an associated vector $v$,

$$\langle \mathrm{Proj}(\theta,v) - v,\ \theta-\theta^* \rangle \geq 0$$

where $\theta^*$ is the true value of the parameter.

- just says that $(\mathrm{Proj}(\theta,v)-v)$ points into the interior of $\Omega_{\Phi(\theta)}$.
- follows from the definition of the projection operator,

$$\langle \mathrm{Proj}(\theta,v)-v,\ \theta-\theta^* \rangle = \begin{cases} 0 & \text{if } \Phi(\theta) < 0 \\ 0 & \text{if } \Phi(\theta) \geq 0 \text{ and } \langle \nabla\Phi, v\rangle < 0 \\ -\Phi(\theta)\left\langle \frac{\nabla\Phi}{\|\nabla\Phi\|}, v\right\rangle \frac{\nabla\Phi}{\|\nabla\Phi\|} & \text{if } \Phi(\theta) \geq 0 \text{ and } \langle \nabla\Phi, v\rangle > 0 \end{cases}$$

Notice that solving for $\Phi(\theta) \leq 1$ leads to

$$\langle \theta, \theta \rangle \leq (1+\epsilon_\theta)\,\theta_{max}^2$$

$\downarrow$

$\epsilon_\theta$ specifies tolerance of $\theta$.

how far beyond $\theta_{max}$ are you willing to go for $\|\theta\|$? About $\sqrt{1+\epsilon_\theta}\,\theta_{max}$.

e.g. $100(\sqrt{1+\epsilon_0} - 1)$ percent beyond limit

What is $-\Psi(\theta)\left\langle \frac{\nabla\Psi}{\|\nabla\Psi\|}, v \right\rangle \frac{\nabla\Psi}{\|\nabla\Psi\|}$

$$\Psi(\theta) = \frac{\langle\theta,\theta\rangle - \theta_{max}^2}{\epsilon_\theta \theta_{max}^2} = \frac{\langle\theta,\theta\rangle}{\epsilon_\theta \theta_{max}^2} - \frac{1}{\epsilon_\theta}$$

$\Rightarrow$

$$\nabla\Psi(\theta) = \nabla\left(\frac{\langle\theta,\theta\rangle}{\epsilon_\theta \theta_{max}^2}\right) = \frac{2\theta}{\epsilon_\theta \theta_{max}^2}$$

took advantage of some $\mathbb{R}^n$ assumptions here.

$\Rightarrow$

$$\frac{\nabla\Psi(\theta)}{\|\nabla\Psi(\theta)\|} = \frac{\theta}{\|\theta\|}$$

$\Rightarrow$

$$-\Psi(\theta)\left\langle \frac{\nabla\Psi}{\|\nabla\Psi\|}, v \right\rangle \cdot \frac{\nabla\Psi}{\|\nabla\Psi\|} = -\Psi(\theta)\left\langle \frac{\theta}{\|\theta\|}, v \right\rangle \frac{\theta}{\|\theta\|}$$

- if projection operator used to define a differential equation, then, for the proper conditions it behaves like $\sigma$-modification. The way we'll use it will be more in line with e-modification.

error & error dynamics

$$e(t) = x(t) - x_m(t)$$

$$\dot{e}(t) = A_m e(t) + b\lambda\left[\Delta k_x^T(t)x(t) + \Delta k_r(t)r(t) - \Delta\alpha(t)\Phi(x(t)) + \epsilon(x(t))\right]$$

with
$$\Delta k_x(t) = k_x(t) - k_x^*$$
$$\Delta k_r(t) = k_r(t) - k_r^*$$
$$\Delta\alpha(t) = \hat{\alpha}(t) - \alpha$$

↳ use projection operator with standard adaptive laws.

$$\dot{k}_x(t) = \Gamma_x \operatorname{Proj}(k_x(t), -x(t)e^T(t)Pb\,\mathrm{sign}(\lambda))$$

$$\dot{k}_r(t) = \gamma_r \operatorname{Proj}(k_r(t), -r(t)e^T(t)Pb\,\mathrm{sign}(\lambda))$$

$$\dot{\hat{\alpha}}(t) = \Gamma_\alpha \operatorname{Proj}(\hat{\alpha}(t), \Phi(x(t))e^T(t)Pb\,\mathrm{sign}(\lambda))$$

Define

$$V(e(t), \Delta k_x(t), \Delta k_r(t), \Delta\alpha(t))$$
$$= e^T(t)Pe(t) + |\lambda|\left[\Delta k_x^T(t)\Gamma_x^{-1}\Delta k_x(t) + \gamma_r^{-1}\Delta k_r^2(t) + \Delta\alpha^T(t)\Gamma_\alpha^{-1}\Delta\alpha(t)\right]$$

↳ where $A_m^T P + PA_m + Q = 0$ for some $Q > 0$.

↓ take time derivative ↓

$$\dot V(t) = -e^T(t)Qe(t) + 2|\lambda|\,\Delta k_x^T(t)\left[x(t)e^T(t)Pb\,\mathrm{sign}(\lambda) + \Gamma_x^{-1}\Delta\dot k_x(t)\right]$$

$$+\ 2|\lambda|\,\Delta k_r(t)\left[r(t)e^T(t)Pb\,\mathrm{sign}(\lambda) + \gamma_r^{-1}\Delta\dot k_r(t)\right]$$

$$+\ 2|\lambda|\,\Delta\alpha^T(t)\left[-\Phi(x(t))e^T(t)Pb\,\mathrm{sign}(\lambda) + \Gamma_\alpha^{-1}\Delta\dot\alpha(t)\right]$$

$$+\ 2e^T(t)Pb\lambda\,\epsilon(x(t))$$

$\Rightarrow$ plug in projected adaptive laws.

- If projection operator does not change its vector argument,

$$\dot V(t) = -e^T(t)Qe(t) + 2e^T(t)Pb\lambda\,\epsilon(x(t))$$

$$\le -\lambda_{min}(Q)\|e(t)\|^2 + 2\|e(t)\|\,\|Pb\|\,|\lambda|\,\epsilon_{max}$$

$(\epsilon_{max} > 0)$

$$\le -\|e(t)\|\left(\lambda_{min}(Q)\|e(t)\| - 2\|Pb\|\,|\lambda|\,\epsilon_{max}\right)$$

$\Rightarrow$

$$\dot V(t) \le 0 \quad \text{if} \quad \|e\| \ge \frac{2|\lambda|\,\|Pb\|\,\epsilon_{max}}{\lambda_{min}(Q)}$$

- If projection operator does change its vector argument,

$$\dot V(t) = -e^T(t)Qe(t) + 2e^T(t)Pb\lambda\,\epsilon(x(t))$$

$$-\ 2|\lambda|\,\Delta k_x^T(t)\,\Phi_x(k_x(t))\frac{\nabla\Phi_x}{\|\nabla\Phi_x\|}\left\langle\frac{\nabla\Phi_x}{\|\nabla\Phi_x\|}, -x(t)e^T(t)Pb\,\mathrm{sign}(\lambda)\right\rangle$$

$$-\ 2|\lambda|\,\Delta k_r(t)\,\Phi_r(k_r(t))\frac{\nabla\Phi_r}{\|\nabla\Phi_r\|}\left\langle\frac{\nabla\Phi_r}{\|\nabla\Phi_r\|}, -r(t)e^T(t)Pb\,\mathrm{sign}(\lambda)\right\rangle$$

$$-\ 2|\lambda|\,\Delta\alpha^T(t)\,\Phi_\alpha(\hat\alpha(t))\frac{\nabla\Phi_\alpha}{\|\nabla\Phi_\alpha\|}\left\langle\frac{\nabla\Phi_\alpha}{\|\nabla\Phi_\alpha\|}, \Phi(x(t))e^T(t)Pb\,\mathrm{sign}(\lambda)\right\rangle$$

$\Rightarrow$

$$\dot{V}(t) = -e^T(t)Qe(t) + 2e^T(t)Pb\lambda\epsilon(x(t))$$

$$- 2e^T(t)Pb\lambda\, \Delta k_x^T(t)\left[\Psi_x(k_x(t))\left\langle \frac{\nabla\Psi_x}{\|\nabla\Psi_x\|}, -x(t)\right\rangle \frac{\nabla\Psi_x}{\|\nabla\Psi_x\|}\right]$$

$$- 2e^T(t)Pb\lambda\, \Delta k_r(t)\left[\Psi_r(k_r(t))\left\langle \frac{\nabla\Psi_r}{\|\nabla\Psi_r\|}, -r(t)\right\rangle \frac{\nabla\Psi_r}{\|\nabla\Psi_r\|}\right]$$

$$- 2e^T(t)Pb\lambda\, \Delta\alpha^T(t)\left[\Psi_\alpha(\hat{\alpha}(t))\left\langle \frac{\nabla\Psi_\alpha}{\|\nabla\Psi_\alpha\|}, \Phi(x(t))\right\rangle \frac{\nabla\Psi_\alpha}{\|\nabla\Psi_\alpha\|}\right]$$

$\Rightarrow$

$$\dot{V}(t) = -e^T(t)Qe(t) + 2e^T(t)Pb\lambda\epsilon(x(t))$$

$$- 2e^T(t)Pb\lambda\, \Psi_x(k_x(t))\left\langle \frac{k_x(t)}{\|k_x(t)\|}, -x(t)\right\rangle \Delta k_x^T(t)\frac{k_x(t)}{\|k_x(t)\|}$$

$$- 2e^T(t)Pb\lambda\, \Psi_r(k_r(t))\left\langle \frac{k_r(t)}{\|k_r(t)\|}, -r(t)\right\rangle \Delta k_r(t)\cdot\frac{k_r(t)}{\|k_r(t)\|}$$

$$- 2e^T(t)Pb\lambda\, \Psi_\alpha(\quad)\left\langle \frac{\hat{\alpha}(t)}{\|\hat{\alpha}(t)\|}, \Phi(x(t))\right\rangle \Delta\alpha^T(t)\frac{\hat{\alpha}(t)}{\|\hat{\alpha}(t)\|}$$

the $\Delta\cdot$ terms are differences. Expand them out and complete the squares. Also replace $\epsilon(x(t))$ with $\epsilon_{max} > 0$ where $\epsilon_{max} > \|\epsilon(x(t))\|$.

$\Rightarrow$

Then can show negative definiteness outside of a compact domain around the origin.

$\Rightarrow$

allows one to then conclude boundedness of signal and to define the bound.

... the rest follows from this ...

The same analysis holds for measurement error:

$$u(t) = k_x^T(t)\,(x(t) + d(t)) + k_r(t)r(t) - \hat{\alpha}^T(t)\,\Phi(x(t)+d(t))$$

$$= k_x^T(t)\,x(t) + k_r(t)r(t) - \hat{\alpha}^T(t)\,\Phi(x(t)) + k_x^T(t)d(t) - \hat{\alpha}^T(t)\,\tilde{\Phi}(x(t),d(t))$$

where $\Phi(x(t)+d(t)) = \Phi(x(t)) + \underbrace{\tilde{\Phi}(x(t),d(t))}_{\text{change induced by measurement error.}}$

$$= k_x^T x(t) + k_r(t)r(t) - \hat{\alpha}^T(t)\,\Phi(x(t)) + \underbrace{\epsilon(k_x(t), x(t), \hat{\alpha}(t))}_{\text{a bit more complicated, but same ideas hold.}}$$

# Disturbance Rejection

- We've already seen this to some degree, but it was either measurement error or parametrization/approximation error of the nonlinearities. What if the disturbances are externally generated? Can we adapt to them? Do we need to?

**Plant:** $\dot{x}(t) = A x(t) + b\lambda\left[u(t) + \alpha^T \Phi(x(t)) + d(t)\right], \quad x(0) = x_0$

**Model:** $\dot{x}_m(t) = A_m x_m(t) + b_m r(t)$

Three realistic classes of disturbances:

$\boxed{1}$ constant, $d(t) = d_0$

$\Rightarrow$ use integral control to remove it.

$\boxed{2}$ vanishing in time, $d(t) \to 0$ as $t \to \infty$.

suppose that $d(t) \in \mathcal{L}_2 \cap \mathcal{L}_\infty$ and $\dot{d}(t) \in \mathcal{L}_\infty$ $\Big\{$ this gives $d(t) \to 0$ as $t \to \infty$ by Barbalat's Lemma.

- we can actually show that $e(t) \to 0$ as $t \to \infty$. Here's how,

Recall that we could get to the point where we had

$$
\begin{aligned}
\dot{V}(t) &= -e^T(t) Q e(t) + 2e^T(t) P b \lambda d(t) \\
&\leq -\lambda_m(Q)\|e(t)\|^2 + 2\|e(t)\| \, \|Pb\| \, |\lambda| \, |d(t)| \\
&\leq -\|e(t)\| \left( \lambda_{min}(Q)\|e(t)\| - 2\|Pb\| \, |\lambda| \, |d(t)| \right)
\end{aligned}
$$

$\Rightarrow$ Assuming no need for projection

$$\dot{V}(t) \leq 0 \quad \text{if} \quad \|e(t)\| \geq \frac{2|\lambda|\,\|Pb\|\,|d(t)|}{\lambda_{min}(Q)}$$

Now, can we go further than this?

Yes, if we try to complete squares

$$\dot{V}(t) \leq -\lambda_{min}(Q)\,\|e(t)\|^2 + \underset{2}{2}\,\|Pb\lambda\|\,\underset{a}{\|e(t)\|}\,\underset{b}{|d(t)|}$$

$\Rightarrow$ $2ab \leq a^2+b^2$

$$\leq -(\lambda_{min}(Q) - c_1^2)\,\|e(t)\|^2 + c_2^2\,|d(t)|^2 \,,$$

where $\lambda_{min}(Q) - c_1 > 0$

$$c_2 = \frac{\|Pb\lambda\|}{c_1}$$

$\Rightarrow$ integrate for all time.

$$V(\infty) - V(0) \leq -(\lambda_{min}(Q) - c_1^2)\int_0^\infty \|e(t)\|^2 dt + c_2^2 \int_0^\infty |d(t)|^2 dt$$

$\Rightarrow$

$$(\lambda_{min}(Q) - c_1)\int_0^\infty \|e(t)\|^2 dt \leq V(0) - V(\infty) + c_2^2 \int_0^\infty |d(t)|^2 dt$$

$\Rightarrow$

$$(\lambda_{min}(Q) - c_1)\int_0^\infty \|e(t)\|^2 dt \leq \underbrace{V(0)}_{\text{bounded}} + c_2^2 \underbrace{\int_0^\infty |d(t)|^2 dt}_{\text{bounded since } d(t) \in \mathcal{L}_2.}$$

$\Rightarrow$

$$\int_0^\infty \|e(t)\|^2 dt \leq \infty$$

$\Rightarrow$

$$e(t) \in \mathcal{L}_2$$

Since all signals $e(t)$, $k_x(t)$, $k_r(t)$, $\hat{\alpha}(t)$, $\dot{k}_x(t)$, $\dot{k}_r(t)$, & $\dot{\hat{\alpha}}(t)$ are bounded, we also get $\dot{e}(t)$ is bounded from error dynamics.

$\Rightarrow$

$$e(t),\ \dot{e}(t) \in \mathcal{L}_\infty \quad \wedge \quad \underline{e(t) \in \mathcal{L}_2}$$

already established.

$\Rightarrow$ Barbalat's Lemma

$$e(t) \to 0 \quad \text{as} \quad t \to \infty.$$

and we are done analyzing case $\boxed{2}$

(last case ...)

$\boxed{3}$ non-vanishing, but uniformly bounded disturbances, $|d(t)| \le d_0 \ \forall t \ge 0$.

We will consider a new control law; one which has an additional parameter that will try to adapt to the bounded disturbance.

Controller: (for case $\boxed{3}$)

$$u(t) = k_x^T(t)x(t) + k_r(t)r(t) - \hat{\alpha}^T(t)\Phi(x(t)) - \beta(t)\tanh\left(\frac{e^T(t)Pb}{\delta}\,\text{sign}(\lambda)\right)$$

$\downarrow$ error dynamics

$$\dot{e}(t) = A_m e(t) + b\lambda\Big[\Delta k_x^T(t)x(t) + \Delta k_r(t)r(t) - \Delta\alpha^T(t)\Phi(x(t)) - \beta(t)\tanh\left(\frac{e^T(t)Pb}{\delta}\,\text{sign}(\lambda)\right)\Big]$$

- use Projection operator-based adaptive laws for $k_x(t)$, $k_r(t)$, $\hat{\alpha}(t)$.

  but what should adaptive law for $\beta(t)$ be?

$$\Delta\dot{\beta}(t) = \dot{\beta}(t) \equiv \gamma_\beta \, \mathrm{Proj}\left(\beta(t),\ e^T(t)Pb \tanh\left(\frac{e^T(t)Pb}{\delta}\,\mathrm{sign}(\lambda)\right)\mathrm{sign}(\lambda)\right)$$

$\Rightarrow$

$$\dot{V}(t) \leq -e^T(t)\,Q\,e(t) + 2|\lambda|\,K\,\delta\,d_0$$

- where we used $\underbrace{0 \leq |\eta| - \eta \tanh\left(\frac{\eta}{\delta}\right) \leq K\delta}$

$K = 0.2785$

this is the bounded term contribution from prior $\dot{V}(t)$ line.

$\Rightarrow$

$$\dot{V}(t) \leq -\lambda_{min}(Q)\,\|e(t)\|^2 + 2|\lambda|\,K\,\delta\,d_0$$

$\Rightarrow$

$$\dot{V}(t) \leq 0 \quad \text{if} \quad \|e(t)\| \geq \sqrt{\frac{2|\lambda|\,K\,\delta\,d_0}{\lambda_{min}(Q)}}$$

$\delta$ scales error ball to be smaller. $\delta \in (0,1]$

and the rest follows from this.

- If Projection operator kicks in, analysis is similar but w/extra terms.

## Composite Adaptation

- The idea behind composite adaptation is to add a predictor to the mix. The role of this additional component to the standard direct MRAC setup should be to improve the adaptive response of the system (given the proper parameter settings).

**Plant :** $\dot{x}(t) = a\,x(t) + b\,u(t)$, $\quad x(0) = x_0$

**Model :** $\dot{x}_m(t) = a_m\,x_m(t) + b_m\,r(t)$, $\quad x_m(0) = x_0$, $\quad a_m < 0$

**Predictor :** $\dot{x}_p(t) = a_p(x_p(t) - x(t)) + a_m\,x(t) + b_m\,r(t)$, $\quad x_p(0) = x_0$, $\quad a_p < 0$

**Controller :** $u(t) = k_x(t)\,x(t) + k_r(t)\,r(t)$

**Adaptation :** → choose your favorite adaptation law here.
for simplicity, will use standard one with augmentation due to the predictor

⇒ model and prediction errors

$$e(t) \equiv x(t) - x_m(t) \quad , \quad \tilde{x}(t) \equiv x_p(t) - x(t)$$

&

$$\dot{e}(t) = a_m\,e(t) + b\Delta k_x(t)\,x(t) + b\Delta k_r(t)\,r(t)$$

$$\dot{\tilde{x}}(t) = a_p\,\tilde{x}(t) + (\underbrace{a_m - a}_{bk_x^*} - bk_x(t))\,x(t) + (\underbrace{b_m}_{bk_r^*} - bk_r(t))\,r(t)$$

$\Rightarrow$ use MATCHING ASSUMPTIONS

$$\dot{\tilde{x}}(t) = a_p \tilde{x}(t) - b\,\Delta k_x(t)\, x(t) - b\,\Delta k_r(t)\, r(t)$$

GREAT!!!

· candidate Lyapunov function

$$V(e(t), \tilde{x}(t), \Delta k_x(t), \Delta k_r(t)) = e^2(t) + \gamma \tilde{x}^2(t) + |b| \left[ \gamma_x^{-1} \Delta k_x^2(t) + \gamma_r^{-1} \Delta k_r^2(t) \right]$$

$\Rightarrow$

$$\begin{aligned}\dot{V}(t) = \; & 2 a_m e^2(t) + 2\gamma a_p \tilde{x}^2(t) \\ & + 2|b|\, \Delta k_x(t) \left[ x(t) e(t) \operatorname{sign}(b) - \gamma x(t) \tilde{x}(t) \operatorname{sign}(b) + \gamma_x^{-1} \Delta \dot{k}_x(t) \right] \\ & + 2|b|\, \Delta k_r(t) \underbrace{\left[ r(t) e(t) \operatorname{sign}(b) - \gamma r(t) \tilde{x}(t) \operatorname{sign}(b) + \gamma_r^{-1} \Delta \dot{k}_r(t) \right]}_{\text{would like for these terms to vanish}}\end{aligned}$$

$$\Delta \dot{k}_x(t) = \dot{k}_x(t) \equiv -\gamma_x x(t) \left[ e(t) - \gamma \tilde{x}(t) \right] \operatorname{sign}(b), \qquad k_x(0) = k_{x,0}$$

$$\Delta \dot{k}_r(t) = \dot{k}_r(t) \equiv -\gamma_r r(t) \left[ e(t) - \gamma \tilde{x}(t) \right] \operatorname{sign}(b), \qquad k_r(0) = k_{r,0}$$

Adaptive Laws

augmentation due to predictor.

$\Rightarrow$

$$\begin{aligned}\dot{V} &= 2 a_m e^2(t) + 2\gamma a_p \tilde{x}^2(t) \\ &= -2|a_m| e^2(t) - 2\gamma |a_p| \tilde{x}^2(t) \le 0 \quad \text{neg. semi-definite}\end{aligned}$$

and the rest follows as before.

Observations / Notes:

1] Setting $\gamma = 0$ reverts to standard direct MRAC structure.

Useful parameter to have; $\gamma$ gives freedom to turn on/off prediction part.

2] Prediction dynamics should be faster than model state dynamics.

Want faster convergence of the predictor than the adaptive tracker.

3] Simulations / empirical evidence suggests better / improved response with this method, in both tracking and initial transient responses during adaptation.