Parameter Convergence & Persistency of Excitation

Consider the scalar, linear case of direct MRAC.

We know that Δk_x & Δk_r are bounded due to stability of the adaptive control system. Can we go any further in the analysis?

fact 1. $e(t) \to 0$ as $t \to \infty$

fact 2. $\dot{e}(t) = a_m e(t) + b(\Delta k_x(t) x(t) + \Delta k_r(t) r(t))$

fact 3. examining the adaptive laws, we see that components of $\dot{e}(t)$ are uniformly cts (due to bdd derivatives).

$\Rightarrow$ Barbalat's Lemma with $e(t)$ & $\dot{e}(t)$

fact 4. $\dot{e}(t) \to 0$ as $t \to \infty$

$\Rightarrow$

$$b(\Delta k_x(t) x(t) + \Delta k_r(t) r(t)) \to 0 \quad \text{as} \quad t \to \infty$$

2 options for achieving this

$\boxed{1}$ $\Delta k_x(t) x(t) \to -\Delta k_r(t) r(t)$ as $t \to \infty$

$\boxed{2}$ $\Delta k_x(t) x(t) \to 0$ and $\Delta k_r(t) r(t) \to 0$ as $t \to \infty$

How can we tell what happens ?

1] this usually happens when $x(t)$ converges to $x_m(t)$ quickly (e.g. - before adaptation is complete)

2] this usually happens when there is sufficient signal to give adaptive process time to converge.

First off, let's examine what happens when given a constant reference signal, $r(t) = r_0$. Due to the fact that the reference model is stable, for large time values,

$$x(t) \cong x_m(t) \cong \frac{b_m}{|a_m|} r_0$$

$\Rightarrow$

$$\Delta k_x(t)\, x(t) + \Delta k_r(t)\, r(t) \cong \Delta k_x(t) \frac{b_m}{|a_m|} r_0 + \Delta k_r(t)\, r_0 \cong 0$$

$\Rightarrow$

$$\Delta k_x(t) \frac{b_m}{|a_m|} + \Delta k_r(t) \cong 0$$

$\Rightarrow$

$$\Delta k_x(t) \frac{b_m}{|a_m|} + \Delta k_r(t) \to 0 \quad \text{as} \quad t \to \infty$$

$\Rightarrow$

$$\Delta k_x(t) \to \frac{a_m}{b_m} \Delta k_r(t) \quad \text{as} \quad t \to \infty$$

either get lucky and both terms ($\Delta k_x(t)$ and $\Delta k_r(t)$) go to zero, or they converge to some steady state error. As indicated by the last equation above, the two errors will be directly proportional.

Now let's examine the case where there is sufficient signal. Actually, really we'll define what that means and show or sketch how that leads to parameter convergence guarantees. Now, for the definition of sufficient signal, we need to define the signal.

Let $v(t) = \begin{bmatrix} x(t) \\ r(t) \end{bmatrix}$, then . . .

Definition. The signal $v(t)$ is said to be persistently exciting if there exist $\alpha > 0$ and $T > 0$ such that for any $t > 0$,

$$\int_t^{t+T} v(\tau) v^T(\tau) d\tau > \alpha \mathbb{1}$$

↑ identity matrix.

- equivalent to asking that $\int_t^{t+T} v(\tau) v^T(\tau) d\tau$ be positive definite, as can be seen by the following

$$\underbrace{z^T \cdot \left(\int_t^{t+T} v(\tau) v^T(\tau) d\tau \right) \cdot z}_{\text{evaluates to a scalar}} > \alpha z^T z \geq 0$$

↑ there is secretely an identity matrix here.

- look up Ioannou & Sun to see their version of it.

- abbreviated as PE

Estimation of the true parameters (and convergence to them) relies on the ability to <u>observe</u> what they should be. How can persistency of excitation be used to conclude observability?

Consider the complete adaptive control system (in the scalar case)

$$\dot{\xi}(t) = A(t)\,\xi(t)$$

$$y(t) = C\,\xi(t)$$

where,

$$\xi(t) = \begin{bmatrix} e(t) \\ \Delta k_x(t) \\ \Delta k_r(t) \end{bmatrix}, \quad A(t) = \begin{bmatrix} a_m & b\,x(t) & b\,r(t) \\ -\gamma_x\, x(t)\,\mathrm{sign}(b) & 0 & 0 \\ -\gamma_r\, r(t)\,\mathrm{sign}(b) & 0 & 0 \end{bmatrix} \qquad (*)$$

$$C = [\,1 \;\; 0 \;\; 0\,]$$

↳ only have access to the error.

Is the system (C, A) observable?

- Due to time dependence, the typical approach using the observability matrix does not work.

$$\Theta = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} \qquad \text{full-rank (column)}$$

We get

$$\Theta = \begin{bmatrix} 1 & 0 & 0 \\ a_m & b\,x(t) & b\,r(t) \\ a_m^2 - b\gamma_x x^2(t)\,\mathrm{sign}(b) - b\gamma_r r^2(t)\,\mathrm{sign}(b) & a_m b\,x(t) & a_m b\,r(t) \end{bmatrix}$$

dependent.

Instead, we will have to rely on the observability Grammian.

$$W_0(t_1,t_2) \equiv \int_{t_1}^{t_2} \Phi^T(\tau;t)\, C^T C\, \Phi(\tau;t)\, d\tau$$

where

$\Phi(t;t_0)$ is the state transition matrix from t_0 to t.

A form of observability known as <u>complete uniform observability</u> can be shown if the observability Grammian is uniformly positive definite, e.g.,

$$\exists\, \alpha>0, T>0 : \forall\, t>0,\; W_0(t,t+T) \geq \alpha \mathbb{1}$$

identity matrix of the proper dimension

- also called <u>uniform complete observability</u>
- notice how this is similar to the PE condition

Determining observability of the time-varying system (*) is complicated by the time-dependence. A transformation of the system will be used instead,

$$(C, A) \text{ observable} \iff (C, A+K(t)C) \text{ observable}.$$

Choose the gain matrix

$$K(t) = \begin{bmatrix} 0 & 0 & 0 \\ \gamma_x x(t) \operatorname{sign}(b) & 0 & 0 \\ \gamma_r r(t) \operatorname{sign}(b) & 0 & 0 \end{bmatrix}$$

leading to

$$\begin{aligned} \dot{\epsilon}(t) &= a_m \epsilon(t) + b v^T(t) \Delta\theta(t) \\ \Delta\dot{\theta}(t) &= 0 \\ y(t) &= \epsilon(t) \end{aligned} \qquad (\dagger)$$

The error dynamics (a_m, b) are stable, [minimum phase].

↑ not so necessary for scalar case, but is for vector case. [holds automatically for scalars]

If $v(t), \dot{v}(t) \in \mathcal{L}_\infty$ and $v(t)$ is PE,

then

$$\hat{v}(t) \equiv \int_{t_0}^{t} e^{a_m(t-\tau)} b\, v(\tau)\, d\tau \quad \text{is PE}.$$

Now, observability condition on the system $(\dagger)$ is equivalent to positive definiteness of

$$\int_t^{t+T} \hat{v}(\sigma) \hat{v}^T(\sigma)\, d\sigma$$

which holds by PE of $\hat{v}(t)$.

$\Rightarrow$

(c, A) is uniformly completely observable.

$\Rightarrow$

convergence to true values is exponentially fast.

In general, it is difficult to assess and validate persistency of excitation.

- For 1^{st} order systems w/ 2 unknowns ← need at least one sinusoidal input.
- For generic linear systems with $2m$ parameters ← need at least m sinusoidal inputs.
- For nonlinear parametrized system → depends on the structure of the system.

Parameter Drift

- We've seen one requirement of the adaptive system if we'd like to gaurantee parameter convergence.
- Here we investigate the opposite: can unknown time-varying components lead to divergence?

Consider the standard scalar setup for direct MRAC, but with a time-varying disturbance

Plant: $\dot{x}(t) = ax(t) + bu(t) \qquad x(0) = x_0$

Reference: $\dot{x}_m(t) = a_m x_m(t) + b_m r(t) \qquad a_m < 0, \; x_m(0) = x_{m,0}$

Control w/Noise: $u(t) = k_x(t)[x(t) + d(t)] + k_r(t) r(t)$

disturbance in measurement of state.

$\Rightarrow$

$$\dot{x}(t) = (a + bk_x(t))\, x(t) + bk_r(t)\, r(t) + bk_x(t)\, d(t)$$

$\Rightarrow$

$$e(t) \equiv x(t) - x_m(t)$$

$$\dot{e}(t) = \dot{x}(t) - \dot{x}_m(t) \qquad e(0) = e_0$$

$$= a_m e(t) + b\Delta k_x(t)\, x(t) + b\Delta k_r(t) r(t) + bk_x(t) d(t)$$

with noise, the adaptive laws become:

$$\left.\begin{aligned} \dot{k}_x(t) &= -\gamma_x (x(t) + d(t))\, e(t)\, \text{sign}(b) \qquad & k_x(0) = k_{x,0} \\ \dot{k}_r(t) &= -\gamma_r\, r(t)\, e(t)\, \text{sign}(b) \qquad & k_r(0) = k_{r,0} \end{aligned}\right\} \text{Adaptation w/Noise}$$

Using the same Lyapunov function as before,

$$V(e(t), \Delta k_x(t), \Delta k_r(t)) = e^2(t) + |b| \left(\gamma_x^{-1} \Delta k_x^2(t) + \gamma_r^{-1} \Delta k_r^2(t) \right)$$

$\Rightarrow$

$$\begin{aligned} \dot{V} = \; & 2 a_m e^2(t) + 2|b| \Delta k_x(t) \left[x(t) e(t) \operatorname{sign}(b) + \gamma_x^{-1} \dot{\Delta k}_x(t) \right] \\ & + 2|b| \Delta k_r(t) \left[r(t) e(t) \operatorname{sign}(b) + \gamma_r^{-1} \dot{\Delta k}_r(t) \right] \\ & + 2 e(t) b k_x(t) d(t) \end{aligned}$$

$$= 2 a_m e^2(t) - 2b \Delta k_x(t) e(t) d(t) + 2b k_x(t) e(t) d(t)$$

$$= 2 a_m e^2(t) + 2 b k_x^* e(t) d(t)$$

$$= 2 e(t) \left[a_m e(t) + b k_x^* d(t) \right]$$

Since we do not know how signs of $e(t)$ and $d(t)$ will relate as functions of time, a conservative estimate will be used to determine negative definiteness. Select / impose the inequality

$$|a_m e(t)| > |b k_x^* d(t)|$$

$$\Rightarrow \quad |e(t)| > \left| \frac{b k_x^*}{a_m} \right| |d(t)|$$

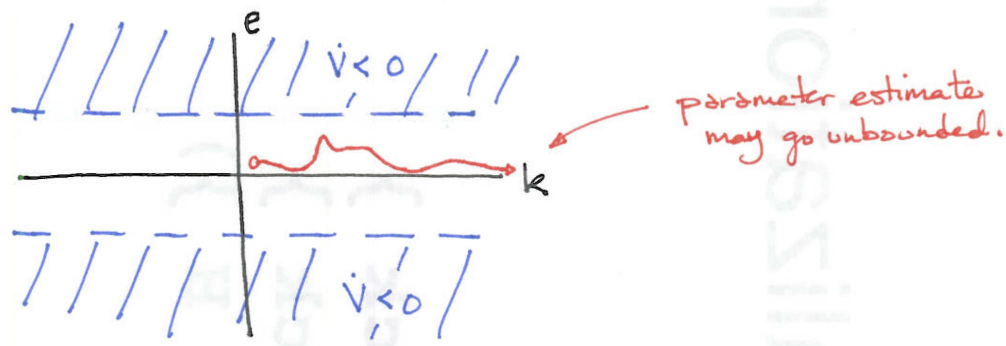
in order to have negative definiteness of $\dot{V}$.

But, once

$$|e(t)| \leq \left| \frac{b k_x^*}{a_m} \right| |d(t)|$$

we can have $\dot{V} \geq 0$.

Visually, we have something along the lines of



where $\dot{V}<0$ for $|e(t)|$ sufficiently large and even $\dot{V}>0$ for $|e(t)|$ sufficiently small.

↳ growth in V may come from diverging parameter values and not growth in $|e(t)|$. There really is not much of a mechanism to prevent parameter drift. If parameter drift goes on long enough, the control effort could go unbounded leading to instability.

(Happens because the set where $\dot{V}>0$ is not compact, hence boundedness can no longer be gauranteed.)

How can we overcome this scenario?

① have persistent excitation.

② modify adaptive laws to ensure that $\dot{V}>0$ only on a compact domain.

Disturbances and Boundedness

With disturbances or other uncertain components, we will no longer be able to show stability using Lyapunov's direct method. Our notion of what's acceptable will have to be weakened to allow for boundedness or (uniform) ultimate boundedness (recall that they are both equivalent, just that one may be easier to prove).

Consider the nonlinear system

$$\dot{x} = f(x,t) \qquad x(0) = x_0 \tag{$*$}$$

where $x \in D \subset \mathbb{R}^n$, D is open and contains the origin, and $f : D \times \mathbb{R} \to \mathbb{R}$ is locally Lipschitz continuous.

In the current investigation, we have a function $V \in C^1(D \times \mathbb{R}; \mathbb{R})$ that does not quite satisfy all the properties we need for stability. The sticking point being that negativity of the time derivative cannot be shown within a compact region about the origin.

We will reexamine the elements of the proof of Lyapunov's direct method to see what adjustments are necessary to conclude something useful.

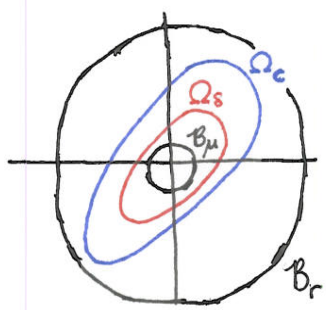
- In what follows we will work as though the Lyapunov function were autonomous, to make life easier (although only minimally). Extension to the time-dependent case follows from the following requirement

$$W_1(x) \le V(x,t) \le W_2(x)$$

for some W_1, W_2 positive definite.

So, we have $\dot{V} < 0$ outside of a compact region about the origin and $\dot{V} ? 0$ inside of this region (could be positive, could be negative, but definitely positive in some cases).

Let B_μ be a ball around the origin containing this region of uncertainty such that $\dot{V}|_{\partial B_\mu} < 0$.

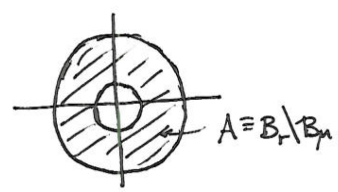


We want to construct the sets visualized to the left where $\overline{B}_\mu \subset \Omega_\delta \subset \Omega_c \subset \overline{B}_r$. Why? Proof of boundedness requires $\overline{B}_\mu$ and $\overline{B}_r$, but method of proof will utilize Ω_δ and Ω_c.

Since we currently have B_μ, goal is to define the rest.

The second thing to do is to assume the existence of a ball B_r (larger than B_μ) with sufficient area inside of the annulus $A \equiv B_r \setminus B_\mu$.

There needs to be enough room to fit the soon to be defined sets Ω_δ and Ω_c inside of A.



What should δ and c be?

Given that $V(x)$ is positive definite, we know that there exist class K functions α_1 and α_2 such that

$$\alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|)$$

With this fact, we know...

- If we define a $c \in (0,\infty)$, we know that

$$x \in \Omega_c \Rightarrow V(x) \leq c \Rightarrow \alpha_1(\|x\|) \leq c \iff \|x\| \leq \alpha_1^{-1}(c).$$

Therefore if we let $c = \alpha_1(r)$, we can ensure $\Omega_c \subset \bar{B}_r$.

- If we have $x \in \bar{B}_\mu$, then we know that

$$x \in B_\mu \Rightarrow \alpha_2(\|x\|) \leq \alpha_2(\mu) \Rightarrow V(x) \leq \alpha_2(\mu).$$

Therefore, if we let $\delta = \alpha_2(\mu)$, we can ensure $\bar{B}_\mu \subset \Omega_\delta$.

- The final step is to go back to r and show how it may be chosen to satisfy the assumptions made. We need to show that $\Omega_\delta \subset \Omega_c$, or equivalently that $\delta < c$.

  Well $\delta < c$ is like $\alpha_2(\mu) < \alpha_1(r)$

  $\Rightarrow$ need r to satisfy $\mu < \alpha_2^{-1}(\alpha_1(r))$

  (or equivalently $r > \alpha_1^{-1}(\alpha_2(\mu))$).

With this construction, any trajectories starting in Ω_c enter Ω_δ in finite time (zero counts too). Once inside of Ω_δ, trajectories don't leave, leading to ultimate boundedness. An ultimate bound can be shown to be $b = \alpha_1^{-1}(\alpha_2(\mu))$.

The complete theorem statement is as follows:

Theorem. Consider the nonlinear system (*). Let $D \subset \mathbb{R}^n$ be a domain containing the origin and $V \in C^1(D \times \mathbb{R}^1; \mathbb{R})$, $\alpha_1, \alpha_2 \in K$, and $W : D \to \mathbb{R}$ positive definite such that

$$\alpha_1(\|x\|) \leq V(x,t) \leq \alpha_2(\|x\|) \qquad \forall x \in D, \ t \in \mathbb{R}^+$$

$$\dot{V}(x,t) \leq -W(x) \qquad \forall x \in D \subset \mathbb{R}^n \ \& \ \|x\| > \mu$$

where

$$\mu < \alpha_2^{-1}(\alpha_1(r))$$

for some r.

Then there exists a class KL function ψ such that for every initial state satisfying $\|x(t_0)\| \leq \mu$, there is a $T = T(x(t_0), \mu) \geq 0$ such that the solution to (*) satisfies

$$\left.\begin{aligned} \|x(t)\| &\leq \psi(\|x(t_0)\|, t - t_0) && \forall \ t_0 \leq t \leq t_0 + T \\ \|x(t)\| &\leq \alpha_1^{-1}(\alpha_2(\mu)) && \forall t \geq t_0 + T \end{aligned}\right\} \quad (\dagger)$$

Moreover, if $D = \mathbb{R}^n$ and $\alpha_1 \in K_\infty$, then $(\dagger)$ holds for any initial state $x(t_0)$, no matter how large μ is.

- Main consideration is $\dot{V}|_A < 0$.

  – Implies $\dot{V} > 0$ must hold only for a compact domain around the origin

(this clearly did not hold in the disturbance example we recently saw)

Deadzone Modification

One thing to do, which is also sometimes done for noisy control systems is to introduce a deadzone for the adaptive laws. This is based on the reasoning that small tracking errors have noise and disturbances as a substantial portion of the signal.

Solution: turn off the adaptation for small tracking errors.

In the SISO case,

$$\dot{k}_x(t) = \begin{cases} -\Gamma_x x(t)\, e^T(t) P b \,\mathrm{sign}(\lambda) & \text{if } \|e\| > \varepsilon \\ 0 & \|e\| \le \varepsilon \end{cases}$$

$$\dot{k}_r(t) = \begin{cases} -\Gamma_r r(t)\, e^T(t) P b \,\mathrm{sign}(\lambda) & \text{if } \|e\| > \varepsilon \\ 0 & \|e\| \le \varepsilon \end{cases}$$

$$\dot{\hat{\alpha}}(t) = \begin{cases} -\Gamma_\alpha \Phi(x(t))\, e^T(t) P b \,\mathrm{sign}(\lambda) & \text{if } \|e\| > \varepsilon \\ 0 & \|e\| \le \varepsilon \end{cases}$$

Of course, in this case ε should be chosen according to the nature of the disturbance (say $\|d(t)\|_\infty$, or an estimate of it), which itself is assumed to not be large.

Let's see what this does.

Plant: $\dot{x}(t) = A x(t) + b\lambda[u(t) + \text{[illegible]}]$

Reference: $\dot{x}_m(t) = A_m x_m(t) + b_m r(t)$

Controller: $u(t) = k_x^T(t)\, x(t) + k_r(t)\, r(t) - \text{[illegible]}$

Adaptation: use adaptive laws w/ deadzone from previous page.

note, that with noise $x(t)$ is $x(t) + d(t)$.

$$V(e(t), \Delta k_x(t), \Delta k_r(t), \Delta\alpha(t)) = e^T(t) P e(t) + |\lambda| \left[\Delta k_x^T(t) \Gamma_x^{-1} \Delta k_x(t) + \gamma_r^{-1} \Delta k_r^2(t) + \text{[illegible]} \right]$$

and the noisy error (using $x(t) + d(t)$)

$$\dot{e}(t) = A_m e(t) + b\lambda[\Delta k_x^T(t) x(t) + \Delta k_r(t) r(t) - \text{[illegible]} + k_x^T(t) d(t)]$$

$\Rightarrow$

$$\begin{aligned} \dot{V} = & -e^T(t) Q e(t) + 2|\lambda|\, \Delta k_x^T(t) \left[x(t)\, e^T(t) P b\, \mathrm{sign}(\lambda) + \Gamma_x^{-1} \dot{\Delta k}_x(t) \right] \\ & + 2|\lambda|\, \Delta k_r(t) \left[r(t)\, e^T(t) P b\, \mathrm{sign}(\lambda) + \Gamma_r^{-1} \dot{\Delta k}_r(t) \right] \\ & + 2\, e^T(t) P b \lambda\, k_x^T(t) d(t) \end{aligned}$$

$2\, k_x^T(t) d(t)\, e^T(t) P b \lambda$

$\Rightarrow$ if $\|e(t)\| > \varepsilon$

$$\begin{aligned} \dot{V} &= -e^T(t) Q e(t) - 2|\lambda|\, \Delta k_x^T(t)\, d(t)\, e^T(t) P b\, \mathrm{sign}(\lambda) + 2\, k_x^T(t) d(t) e^T(t) P b \lambda \\ &= -e^T(t) Q e(t) - 2 (k_x^*)^T d(t)\, e^T(t) P b \lambda \end{aligned}$$

is negative semi-definite for ε large enough.

Plant: $\dot{x}(t) = A\,x(t) + b\lambda\,u(t)$

Reference: $\dot{x}_m(t) = A_m\,x_m(t) + b_m r(t)$

Controller: $u(t) = k_x^T(t)\,x(t) + k_r(t)\,r(t)$

Adaptation: use adaptive laws with deadzone from the previous page.

$$V(e(t), \Delta k_x(t), \Delta k_r(t), \Delta\alpha(t)) = e^T(t) P e(t) + |\lambda| \left[\Delta k_x^T(t) \Gamma_x^{-1} \Delta k_x(t) + \gamma_r^{-1} \Delta k_r^2(t) \right]$$

and the noisy error dynamics with $x(t) \mapsto x(t) + d(t)$

$$\dot{e}(t) = A_m e(t) + b\lambda \left[\Delta k_x^T(t) x(t) + \Delta k_r(t) r(t) + k_x^T(t) d(t) \right]$$

$\Rightarrow$

$$\begin{aligned}\dot{V}(t) = & -e^T(t) Q e(t) + 2|\lambda|\,\Delta k_x^T(t) \left[x(t) e^T(t) P b\, \mathrm{sign}(\lambda) + \Gamma_x^{-1} \Delta\dot{k}_x(t) \right] \\ & + 2|\lambda|\,\Delta k_r(t) \left[r(t) e^T(t) P b\, \mathrm{sign}(\lambda) + \gamma_r^{-1} \Delta\dot{k}_r(t) \right] \\ & + 2 k_x^T(t) d(t)\, e^T(t) P b \lambda\end{aligned}$$

$\Rightarrow$

if $\|e(t)\| > \varepsilon$, then

$$\dot{V}(t) = -e^T(t) Q e(t) - 2|\lambda|\,\Delta k_x^T(t) d(t)\, e^T(t) P b\, \mathrm{sign}(\lambda) + 2 k_x^T(t) d(t)\, e^T(t) P b \lambda$$

$$= -e^T(t) Q e(t) - 2 (k_x^*)^T d(t)\, e^T(t) P b \lambda$$

it is negative semi-definite for ε large enough.

Now, if $\|e(t)\| < \varepsilon$, then

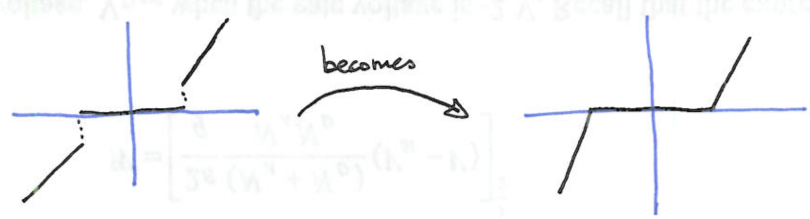
$$\dot{V}(t) = -e^T(t)Qe(t) + 2\Delta k_x^T(t)\,x(t)\,e^T(t)Pb\lambda + 2\Delta k_r(t)\,r(t)\,e^T(t)Pb\lambda + 2k_x^T(t)\,d(t)\,e^T(t)Pb\lambda$$

$$= -e^T(t)Qe(t) + 2e^T(t)Pb\lambda\left[\Delta k_x^T(t)x(t) + \Delta k_r(t)r(t) + k_x^T(t)d(t)\right]$$

no gaurantees on what happens.

- the deadzone modification introduces discontinuities, which is not good from an existence & uniqueness perspective. May introduce switching like in problem 10, homework 1.

- to make deadzone continuous (Lipschitz) modify it to remove the discontinuities,



(see Ioannu & Sun for more details)

The basic adjustments is (for k_x):

$$k_x(t) = \begin{cases} -\Gamma_x x(t)\left(e(t) - \dfrac{\varepsilon e(t)}{\|e(t)\|}\right)^T Pb\,\mathrm{sign}(\lambda) & \text{if } \|e(t)\| > \varepsilon \\ 0 & \text{otherwise} \end{cases}$$

Other Methods:

σ-modification $\rightarrow$ adds an additional damping term to keep the adaptive parameters bounded

$$\dot{k}_x(t) = -\Gamma_x\left[x(t)\, e^T(t) P b \,\mathrm{sign}(\lambda) + \sigma_x k_x(t)\right]$$

$$\dot{k}_r(t) = -\gamma_r\left[r(t)\, e^T(t) P b \,\mathrm{sign}(\lambda) + \sigma_r k_r(t)\right]$$

$$\dot{\hat{\alpha}}(t) = \Gamma_\alpha\left[\Phi(x(t))\, e^T(t) P b \,\mathrm{sign}(\lambda) - \sigma_\alpha \hat{\alpha}(t)\right]$$

e-modification $\rightarrow$ also adds damping but with an error-scaled factor included.

$$\dot{k}_x(t) = -\Gamma_x\left[x(t) e^T(t) P b \,\mathrm{sign}(\lambda) + \|e^T(t) P b\|\, \sigma_x k_x(t)\right]$$

$$\dot{k}_r(t) = -\gamma_r\left[r(t)\, e^T(t) P b \,\mathrm{sign}(\lambda) + \|e^T(t) P b\|\, \sigma_r k_r(t)\right]$$

$$\dot{\hat{\alpha}}(t) = \Gamma_\alpha\left[\Phi(x(t))\, e^T(t) P b \,\mathrm{sign}(\lambda) - \|e^T(t) P b\|\, \sigma_\alpha \hat{\alpha}(t)\right]$$

- The "damping" terms created the compact set required to prove boundedness.

  The bounds are affected by the choice of σ (adaptation also affected by σ).