Investigating Stability

Linear control theory sought to analyze the control system

$$\dot{x} = Ax + Bu, \qquad x(t_0) = x_0 .$$

Stability analyses of the plant only

$$\dot{x} = Ax, \qquad x(t_0) = x_0$$

involved analysis of the eigenvalues, λ, of A.

- stability required $Re(\lambda) \le 0$ for all λ of A.
- asymptotic stability required $Re(\lambda) < 0$ for all λ of A.

  for linear systems asymptotic stability $\Leftrightarrow$ exponential stability
- instability resulted if there existed a λ of A with $Re(\lambda) > 0$.

→ Since many linear plants are derived from nonlinear versions, does the same conclusion hold for the ~~linear approximation~~ true nonlinear system given the linear approximation?

eg. When is the following procedure valid? or How is it valid?

1) find equilibria.
2) linearize about equilibria.
3) check linearized system for stability ($Re(\lambda) < 0$).
4) arrive at appropriate conclusion.

Theorem [Lyapunov's Indirect Method]. Let x_e be an equilibrium of the system

$$\dot{x} = f(x), \qquad x(0) = x_0$$

where f is C^1 in a neighborhood $N(x_e) \subset \mathbb{R}^n$. Define

$$A \equiv Df(x_e).$$

Then, at x_e, the original system is

1. locally asymptotically stable if $Re(\lambda) < 0$ for all λ of A.
2. unstable if $Re(\lambda) > 0$ for at least one λ of A.

* no conclusion can be drawn if $\exists \lambda : Re(\lambda) = 0$, so long as no λ are such that $Re(\lambda) > 0$.

- see Khalil for proof, or other textbook on nonlinear systems analysis.

* A natural question to ask is: Why not conclude local exponential stability if $Re(\lambda) < 0$ for all λ of A?

- This does in fact hold, but is not part of the original statement. See Khalil's book, or another equivalent book on nonlinear dynamical systems theory / analysis.

Example. (Pendulum)

The pendulum dynamics are:

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -\frac{d}{mr} x_2 - \frac{g}{r^2}\sin(x_1) = -b x_2 - a\sin(x_1)$$

for $b = \frac{d}{mr} \geq 0$ and $\frac{g}{r^2} > 0$. (for $b=0$, there is no damping)

The equilibria are $x_e = (2k\pi, 0)$ and $((2k+1)\pi, 0)$ $k \in \mathbb{Z}$

swing down swing up

1) linearize swing down

$$A = Df(x_e) = \begin{bmatrix} 0 & 1 \\ -a\cos(x_{e,1}) & -b \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -a & -b \end{bmatrix}$$

$\Rightarrow$

$$\lambda = -\frac{b}{2} \pm \frac{\sqrt{b^2 - 4a}}{2}$$

no damping $(b=0)$ $\Rightarrow$ $\mathrm{Re}(\lambda) = 0$ NO CONCLUSION!

damping $(b>0)$ $\Rightarrow$ $\mathrm{Re}(\lambda) < 0$ LOCAL EXP. STABILITY.

2) linearize swing up

$$A = Df(x_e) = \begin{bmatrix} 0 & 1 \\ -a\cos(x_{e,1}) & -b \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ a & -b \end{bmatrix}$$

$\Rightarrow$

$$\lambda = -\frac{b}{2} \pm \frac{\sqrt{b^2 + 4a}}{2}$$

irrespective of b, there is a λ such that $\mathrm{Re}(\lambda) > 0$. UNSTABLE.

Note

- Theorem required $f \in C^1$ and not just Lipschitz continuity.
- Results are local
- Sometimes may not be able to conclude anything.

Questions:

- Are there more global tests?
- Can one use them to quantify the stability region (e.g., domain of attraction)?
- Are they applicable to nonlinear systems?

Answers:

- Yes, yes, & yes, but these tests come with their own strengths and weaknesses.

1) Lyapunov stability theorems
2) La Salle's Invariance Principle | Invariant Set theorems

Lyapunov Stability Theorems

Definition. A continuous function $V: \mathbb{R}^n \to \mathbb{R}$ is positive definite on a domain $D \subset \mathbb{R}^n$ if

$$V(x) > 0 \quad \forall x \in D \setminus \{0\} \quad \text{and} \quad V(0) = 0.$$

Definition. A continuous function $V: \mathbb{R}^n \to \mathbb{R}$ is positive semidefinite on a domain $D \subset \mathbb{R}^n$ if

$$V(x) \geq 0 \quad \forall x \in D.$$

- if inequality reversed get negative (semi) definite.

Theorem [Lyapunov's Direct Method]. Let $x=0$ be an equilibrium point for

$$\dot{x} = f(x), \qquad x(0) = x_0 \quad , \quad x \in \mathbb{R}^n$$

for which f is Lipschitz continuous in $D \subset \mathbb{R}^n$ [redacted], where D contains the origin. Suppose that there exists a $V \in C^1(D;\mathbb{R})$ positive definite in D such that

$$\dot{V}(x(t)) \leq 0 \qquad \forall x(t) \in D.$$

Then the equilibrium point is stable.

If

$$\dot{V}(x(t)) < 0 \qquad \forall x(t) \in D \setminus \{0\},$$

then it is asymptotically stable.

Furthermore, if there are positive scalars $k_1, k_2, k_3 > 0$ and a $p \geq 1$ such that

$$k_1 \|x\|^p \leq V(x) \leq k_2 \|x\|^p \qquad \forall x \in D$$

and

$$\dot{V}(x(t)) \leq -k_3 V(x(t)) \qquad \forall x(t) \in D,$$

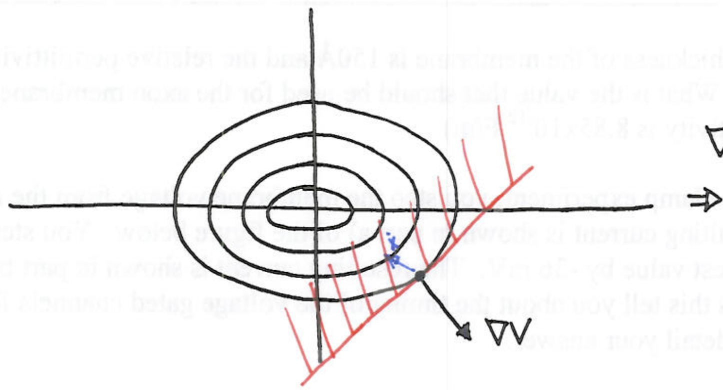
then it is exponentially stable.

proof.

1] First, the touchy-feely version, then the actual version

Take the level sets of V and examine what $\dot{V}(x(t)) = DV(x(t)) \cdot \dot{x}(t)$ means.

$$= \nabla V \cdot \dot{x} \leq 0$$



$\Rightarrow$ $\dot{x}$ must point in the direction of the red hatched half-plane.

$\Rightarrow$ $\dot{x}$ leads inside to a lower level set, or it follows along an isocontour (eg. $V(x) = c$).

Thus since solutions must lie along or inside $V(x_0) = c$, the system is stable.

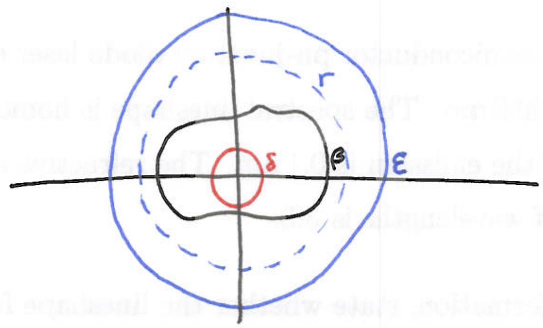
2] Now, the proof will essentially take an ϵ-neighborhood of the origin and construct a δ-neighborhood for it to show stability. Sandwhiched between these two sets will be another set based on the Lyapunov function. Define Ω_β,

$$\Omega_\beta = \{ x \in \mathbb{R}^n \mid V(x) \leq \beta \}.$$

The goal is to show that the following construction of sets is possible

$$N(\delta) \subset \Omega_\beta \subset N(r) \subset N(\epsilon) \, ,$$

visualized as follows



Taking advantage of the fact that trajectories starting in Ω_β stay in Ω_β (they are invariant) leads to the conclusion of stability.

- Start with some $\epsilon > 0$ and find an $r \in (0, \epsilon]$ such that $N(r) \subset D$.

- Next, define $\alpha = \min_{\|x\|=r} V(x) > 0$. Select a $\beta \in (0, \alpha)$ such that the set $\Omega_\beta \subset N(r)$. It will contain the origin by virtue of V being positive definite.

  Also, by construction, solution curves ~~lying in~~ beginning in Ω_β stay in Ω_β

  because: $\frac{d}{dt} V(x(t)) = \dot{V}(x(t)) \leq 0 \Rightarrow V(x(t)) \leq V(x(0)) \leq \beta$.

- Continuity of V means that for β, there exists a δ such that

$$\|x - 0\| < \delta \Rightarrow \| V(x) - V(0) \| < \beta$$

$$\Rightarrow$$

$$\|x\| < \delta \Rightarrow \| V(x) \| < \beta$$

and the great thing is that ~~the~~ ~~the~~ the δ-neighborhood of the origin $N(\delta)$ lies inside of Ω_β.

- Consequently trajectories starting in $N(\delta)$ also start in Ω_β, and, therefore, ~~never~~ are completely contained with Ω_β.

$\Rightarrow$

We have constructed: $N(\delta) \subset \Omega_\beta \subset N(r) \subset N(\epsilon)$

such that $x(0) \in N(\delta) \Rightarrow x(0) \in \Omega_\beta \Rightarrow x(t) \in \Omega_\beta \Rightarrow x(t) \in N(\epsilon)$.

3] Now, we will show the asymptotic stability part.

Let $x(t)$ be a solution starting in $N(\delta)\setminus\{0\} \subset \Omega_\beta\setminus\{0\}$.

Because Ω_β is compact, there exists a sequence of times $\{t_n\}_1^\infty$ such that $t_n \to \infty$ as $n \to \infty$ where it also holds that $x(t_n) \to z_0 \in \Omega_\beta$ as $n \to \infty$, for some z_0.

We assert that $z_0 = 0$, and show by contradiction.

V continuous means that if $x(t_n) \to z_0$ as $n \to \infty$, then $V(x(t_n)) \to V(z_0)$ as $n \to \infty$.

$V(x(t))$ decreasing only means that $V(x(t)) > V(z_0)$ (approaches from above).

If $z_0 \neq 0$, then ~~any~~ the solution $z(t)$ starting at z_0 must satisfy

$$V(z(t)) < V(z_0) \quad \forall\, t > 0$$

Choose a y_0 close to z_0. If close enough, the following inequality

$$V(y(T)) < V(z_0)$$

must hold for the solution $y(t)$ starting at y_0, for some time $T > 0$.

Good, so then pick $y_0 = x(t_n)$ for an n sufficiently large, for this will yield the contradiction

$$V(y_0(T)) = V(x(t_n+T)) < V(z_0).$$

Therefore $z_0 = 0$ must hold.

Basically we just showed that the only possible limit point of the set $\{x(t) \mid t \geq 0\}$ is the origin (and it must have one!).

4] Lastly, the exponentially stability conclusion must be proven,

$$\dot{V}(x(t)) \leq -k_3 V(x(t))$$

$\Rightarrow$ comparison lemma

$$V(x(t)) \leq V(x(0))\, e^{-k_3 t}$$

$\Rightarrow$

$$k_1 \|x(t)\|^p \leq V(x(t)) \leq V(x(0))\, e^{-k_3 t} \leq k_2 \|x(0)\|^p e^{-k_3 t}$$

$\Rightarrow$

$$\|x(t)\|^p \leq \frac{k_2}{k_1} \|x(0)\|^p e^{-k_3 t}$$

$\Rightarrow$

$$\|x(t)\| \leq \left(\frac{k_2}{k_1}\right)^{1/p} \|x(0)\|\, e^{-\frac{k_3}{p} t}$$

and that's it!

Comments on Lyapunov's Direct Method

1) Don't need the integral curves of f (e.g. solution trajectories)

Only need to know behavior of V given f.

2) Sufficient but not necessary!

You can't find a V, you can't conclude anything.

(but there are negative versions to show instability in case that's really how it goes. They too are sufficient but not necessary.)

3) The big problem is discovering V.

Energy is a great option for mechanical systems.

There are some methods for finding V.

There are some converse Lyapunov theorems stating when such a V must exist.

4) As stated, the theorem cannot be used to infer global stability.

Both Khalil and Hovakimyan's lecture notes cover a Lyapunov function whereby global stability cannot be determined. The failure lies in Ω_β. Since global mean that the stability conditions must hold for arbitrary δ, this means we have to have Ω_β stay compact for arbitrary β.

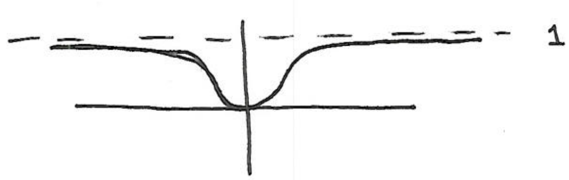
Recalling the sets, $\quad N(\delta) \subset \Omega_\beta \subset N(r) \subset N(\epsilon)$

if δ too big, then ~~β~~ Ω_β may go unbounded.

The example for this is

$$V(x_1, x_2) = \frac{x_1^2}{1+x_1^2} + x_2^2$$

for $x_2 = 0$, the ~~solution~~ ~~evaluation~~ plot along the x_1-axis of $V(x_1, 0)$ is



thus we get $\quad \Omega_\beta$ bounded if $\beta < 1$

and $\quad \Omega_\beta$ unbounded if $\beta > 1$

This would be a poor function for deciding/proving stability since a function starting in Ω_β can remain in Ω_β but still escape to infinity. That's not good. We want V to get arbitrarily large as $\|x\|$ gets arbitrarily large (more or less).

A more technical way of putting this is the following:

Definition. A function $V : \mathbb{R}^n \to \mathbb{R}$ such that $V(x) \to \infty$ as $\|x\| \to \infty$ is called <u>radially unbounded</u>.

Theorem. Let $x = 0$ be an equilibrium point for

$$\dot{x} = f(x), \quad x(0) = x_0, \quad x \in \mathbb{R}^n,$$

for which f is Lipschitz continuous on [redacted] $\mathbb{R}^n$. Suppose there exists a radially unbounded $V \in C^1(\mathbb{R}^n; \mathbb{R})$ positive definite in $\mathbb{R}^n$ such that

$$\dot{V}(x(t)) < 0 \qquad \forall\, x(t) \neq 0,$$

then the equilibrium point is globally asymptotically stable.

- Can conclude ~~the~~ likewise for global exponential stability under the proper additional conditions.

Examples.

1]
$$\dot{x}_1 = x_2 - a x_1^3 \qquad\qquad a>0,\ b>0$$
$$\dot{x}_2 = -x_1 - b x_2$$

candidate Lyapunov function

$$V(x) = \tfrac{1}{2}x_1^2 + \tfrac{1}{2}x_2^2 > 0 \qquad \text{for } x \neq 0$$

Test negative (semi-) definiteness of derivative

$$\dot{V}(x(t)) = x_1\dot{x}_1 + x_2\dot{x}_2 = x_1(x_2 - a x_1^3) + x_2(-x_1 - b x_2)$$
$$= -a x_1^4 - b x_2^2 < 0 \qquad \text{for } x \neq 0$$

definite!

$\Rightarrow$ asymptotic stability

$\Rightarrow$ V is radially unbounded

global asymptotic stability

2]
$$\dot{x}_1 = \cancel{-x_1 + 4x_2} = -x_1 + 4x_2$$
$$\dot{x}_2 = \cancel{-x_1 - x_2^3} = -x_1 - x_2^3$$

pick a candidate Lyapunov function w/ some design freedom

$$V(x) = \cancel{\tfrac{1}{2}x_1^2 + \tfrac{1}{2}ay^2}\ \tfrac{1}{2}x_1^2 + \tfrac{1}{2}a x_2^2$$

check negative (semi) definiteness

$$\dot{V}(x(t)) = x_1\dot{x}_1 + a x_2\dot{x}_2 = x_1(-x_1 + 4x_2) + a x_2(-x_1 - x_2^3)$$
$$= -x_1^2 + 4x_1x_2 - a x_1x_2 - a x_2^4$$
$$= -x_1^2 + (4-a)x_1x_2 - a x_2^4$$

$\Rightarrow$ let $a = 4$

$$\dot{V}(x(t)) = -x_1^2 - 4x_2^4 < 0 \qquad \text{for } x \neq 0$$

$\Rightarrow$ radially unbounded V

globally asymptotically stable

$\boxed{3}$

$$\dot{x} = -x + 2y^3 - 2y^4$$
$$\dot{y} = -x - y + xy$$

this one is a bit more complex, so we pick a candidate Lyapunov function with more parameters, and hope for the best,

$$V(x,y) = x^m + ay^n .$$

$\Rightarrow$

$$\begin{aligned}
\dot{V} &= mx^{m-1}\dot{x} + any^{n-1}\dot{y} \\
&= mx^{m-1}(-x+2y^3-2y^4) + any^{n-1}(-x-y+xy) \\
&= -mx^m + 2mx^{m-1}y^3 - 2mx^{m-1}y^4 - anxy^{n-1} - any^n + anxy^n \\
&= -mx^m - any^n + 2mx^{m-1}y^3 - anxy^{n-1} - 2mx^{m-1}y^4 + anxy^n
\end{aligned}$$

choosing $m = 2$ and $n = 4$ allows terms to be factored

$$= -2x^2 - 4ay^4 + (4 - 4a)xy^3 + (-4 + 4a)xy^4$$

let $a = 1$

$$= -2x^2 - 4y^4 < 0$$

$\Rightarrow$

for $V(x,y) = x^2 + y^4$ we can show global asymptotic stability.

Just in case you need to prove instability, here are a couple of negative Lyapunov theorems.

Theorem. Let $x=0$ be an equilibrium point for

$$\dot{x} = f(x), \qquad x(0) = x_0, \quad x \in \mathbb{R}^n,$$

for which f is Lipschitz continuous on $D \subset \mathbb{R}^n$, where D contains the origin. Suppose there exists a $V \in C^1(D;\mathbb{R})$ positive ~~positive~~ definite in D such that

$$\dot{V}(x(t)) > 0 \qquad \forall\, x(t) \in D \setminus \{0\},$$

then the equilibrium point is unstable.

- This is a bit restrictive since it defines a function V whose derivative is positive definite in $D\setminus\{0\}$. Since instability really just needs one example of a trajectory that leaves some ϵ-neighborhood when beginning in some arbitrary δ-neighborhood.

- Can we define a subset $\Omega \subset D$ where a properly defined ~~subset~~ Lyapunov function can show instability?

Theorem [Chaetev's Theorem]. Let $x=0$ be an equilibrium point of

$$\dot{x} = f(x), \quad x(0) = x_0, \quad x \in \mathbb{R}^n,$$

where f is Lipschitz continuous on $D \subset \mathbb{R}^n$ where D contains the origin. Let $V \in C^1(D;\mathbb{R})$ be such that

$$V(0) = 0 \quad \text{and} \quad \forall\, \delta > 0,\ \exists\, x_0 \in N(0,\delta) : V(x_0) > 0.$$

Define the set Ω to be

$$\Omega \equiv \{ x \in N(0,\delta) \mid V(x) > 0 \}$$

and suppose that

$$\dot{V}(x(t)) > 0 \qquad \forall\, x(t) \in \Omega.$$

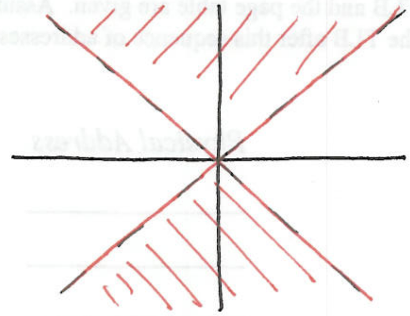
Then the equilibrium point is unstable.

* Basically the region Ω ~~acts like a regio~~ contains trajectories that move away from the origin if they start in Ω. Thus Ω can be used to define an ϵ-neighborhood with a trajectory beginning in any δ-neighborhood such that it escapes the ϵ-neighborhood at some point in time.

~~The region Ω is designed~~

The Lyapunov function V and its associated Ω are such that trajectories starting in Ω stay in Ω.

An example is $V(x) = \frac{1}{2}(x_2^2 - x_1^2)$



in the shaded region it is positive definite (region includes origin)

If it is possible to show that $\dot{V}(x(t)) > 0$ in ~~the~~ any connected subset of the shaded region containing the origin, then the origin is unstable.

$$\dot{x}_1 = -x_1^3 + x_2$$
$$\dot{x}_2 = x_1^6 - x_2^3$$

1. find equilibria:

$$0 = -x_1^3 + x_2$$
$$0 = x_1^6 - x_2^3$$

$\Rightarrow$

$x_2 = x_1^3$ & $x_2^3 = x_1^6$

$$(x_1^3)^3 = x_1^6$$

$x_1^9 \neq x_1^6$ if $x_1 \neq 0, 1$

$\Rightarrow$

only solutions are $x_e = (0,0)$ and $x_e = (1,1)$

2. assess stability

for (1,1), linearization can be used to show local stability.

$$A_{(1,1)} = \left.\frac{\partial f}{\partial x}\right|_{x_e=(1,1)} = \begin{bmatrix} -3x_1^2 & 1 \\ 6x_1^5 & -3x_2^2 \end{bmatrix}_{x_e=(0,0)} = \begin{bmatrix} -3 & 1 \\ 6 & -3 \end{bmatrix}$$

$\Rightarrow$

$$\lambda = -3 \pm \sqrt{8}$$

$\Rightarrow$

$$Re(\lambda) < 0$$

$\Rightarrow$

local stability

for (0,0), linearizations says nothing since $A_{(0,0)} = \left.\frac{\partial f}{\partial x}\right|_{x_e=(0,0)} = 0$.

must use more direct method. The system is, in fact, unstable at the origin. This can be shown in a tricky way using nullclines. Nullclines are surfaces in the space where a component of the dynamics vanishes. For 2D systems, these are lines.

$$\text{if} \quad \underset{\text{at a point}}{x_2 = x_1^3}, \quad \text{then} \quad \dot{x}_1 = 0 \quad \text{and} \quad \dot{x}_2 = x_1^6 - x_1^9$$

$\Rightarrow$

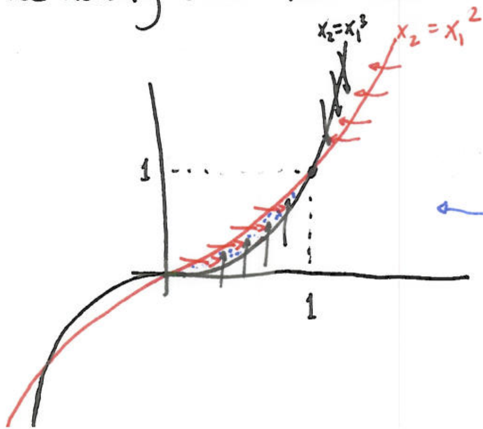
$$\left.\begin{array}{ll} \dot{x}_2 < 0 & \text{if } x_1 > 1 \\ \dot{x}_2 \geq 0 & \text{if } x_1 \leq 1 \end{array}\right\} \text{ trajectory has only vertical component.}$$

$$\text{if} \quad \underset{\text{at a point}}{x_2 = x_1^2}, \quad \text{then} \quad \dot{x}_1 = x_1^2 - x_1^3 \quad \text{and} \quad \dot{x}_2 = 0$$

$\Rightarrow$

$$\left.\begin{array}{ll} \dot{x}_1 < 0 & \text{if } x_1 > 1 \\ \dot{x}_1 \geq 0 & \text{if } x_1 \leq 1 \end{array}\right\} \text{ trajectory has only horizontal component}$$

Let's see visually what this means



← trajectories enter the blue-dotted region between the two nullclines and never leave it.

if we define D to be the region enclosed by the nullclines

$$D = \{ (x_1, x_2) \mid 0 \le x_1 < 1 , \; 0 \le x_2 < 1 , \; x_1^3 \le x_2 \le x_1^2 \}$$

and define the function

$$V = \frac{1}{2}(x_1^2 + x_2^2) \geq 0 \quad \text{on } D$$

$\Rightarrow$

$$\dot{V} = x_1 \dot{x}_1 + x_2 \dot{x}_2$$

$$= x_1 (x_2 - x_1^3) + x_2 (x_1^6 - x_2^3)$$

$\Rightarrow$ on D $\quad x_2 \geq x_1^3$ and $x_1^6 \geq x_2^3$

$$\dot{V} \geq 0 \quad \text{on } D$$

equality only holds for $(0,0)$ and $(1,1)$.

but this is not included in D.

$\Rightarrow$

$$\Omega = \{ (x_1, x_2) \in N(0, \delta) \subset D \mid V(x_1, x_2) > 0 \} \quad \text{for } \delta < 1 .$$

well, on Ω we have $\dot{V}|_{\Omega} > 0$

$\Rightarrow$ Chaetev's Theorem.

instability.