Stability of Dynamical Systems

Want to continue analysis of the initial value problem,

$$\dot{x} = f(x,t) \quad , \quad x(0) = x_0 \qquad (*)$$

In particular, we want to analyze system trajectories and characterize them if possible. Implicit is the assumption that system trajectories exist & are unique.

First, some notation:

if $\dot{x} = f(x)$ the system is autonomous

if $\dot{x} = A(t)x$ the system is linear

if $\dot{x} = Ax$ the system is linear autonomous

Definition. A state x_e is an equilibrium of $(*)$, if once $x(t') = x_e$, then $x(t) = x_e$ for all $t \geq t'$ on the maximal interval of existence of the solution

Definition. A equilibrium x_e is an isolated equilibrium if $\exists r > 0$ such that the neighborhood $N(x_e; r)$ has no other equilibria.

- An equivalent characteristic of an equilibrium x_e to the definition above is the condition

$$f(x_e, t) \equiv 0 \qquad \forall\, t \geq t_0 .$$

Examples of Equilibria:

1]
$$\dot{x}_1 = -x_1 + 2$$
$$\dot{x}_2 = -x_2^3 + x_1 x_2$$

$\Rightarrow$ solve for $f(x) = 0$,

$$0 = -x_1 + 2$$
$$0 = -x_2^3 + x_1 x_2 = x_2(x_1 - x_2^2)$$

$\Rightarrow$

$$x_1 = 2 \quad \text{and} \quad \begin{cases} x_2 = 0 \\ x_1 - x_2^2 = 0 \end{cases}$$

$\Rightarrow$

$$x_1 = 2 \quad \text{and} \quad x_2 = 0, \sqrt{2}, -\sqrt{2}$$

$\Rightarrow$

$(2,0)$, $(2,\sqrt{2})$, and $(2,-\sqrt{2})$ are equilibria.
(they are isolated)

2] Linear systems

$$\dot{x} = Ax$$

What are the equilibria?

a) ~~either~~ if A is 1-1, then only the origin is an equilibrium state.

b) if $\text{null}(A) \neq \{0\}$, then there are an ∞ amount of equilibria.

example: $\ddot{x} + \dot{x} = 0$

$\hookrightarrow$ $x_1 = x$, $x_2 = \dot{x}$

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -x_2$$

$\hookrightarrow$ $\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} x$

rank is 1.

$\Rightarrow$ equilibria

$x_1 =$ anything, $x_2 = 0$

3]

$$\dot{x}_1 = x_1 x_2$$
$$\dot{x}_2 = x_1^2$$

$\Rightarrow$ equilibria

$$0 = x_1 x_2$$
$$0 = x_1^2$$

$\Rightarrow$

$x_1 = 0$, x_2 = anything

the whole x_2-axis consists of equilibria.

(none of them are isolated)

4]

$$\dot{x} = \sin(x)$$

$\Rightarrow$

equilibria are $x_e = k\pi$ for $k \in \mathbb{Z}$

(they are isolated)

↳ here, we have a countable ∞ of isolated equilibria

5]

$$\dot{x} = x^2 + 5$$

$\Rightarrow$ equilibria

there are none.

note:

linear autonomous system	nonlinear system
only one equilibrium or a cts ∞ of them	no equilibria, one equilibria or a finite amount, a countable ∞, or an uncountable ∞ (eg a continuum).

The next question to ask is: What happens local to an equilibrium point? what if we are somewhere close to it?

- the following definitions attempt to characterize what can happen. The goal after this is to see if we can prove that something does in fact happen!

Definition. The equilibrium state x_e is stable (in the sense of Lyapunov) if, for arbitrary t_0 and $\epsilon > 0$, $\exists\, \delta(\epsilon, t_0) : \|x_0 - x_e\| < \delta$ implies that $\|x(t; x_0, t_0) - x_e\| < \epsilon \quad \forall\, t \geq t_0$. Sometimes called Lyapunov stable.

Definition. The equilibrium state x_e is unstable if it is not stable.

- Mathematically, we mean that

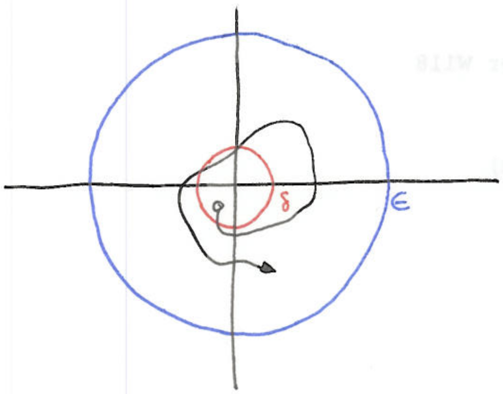
$$\exists\, \epsilon > 0 : \forall\, \delta > 0,\ \exists\, x_0, T(\delta) > t_0 : \ \|x_0 - x_e\| < \delta \text{ and}$$

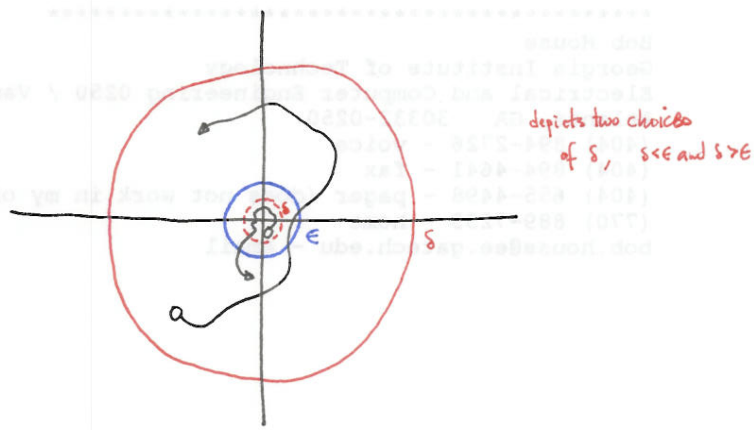
- OK, so what do these mean in English? How can I visualize the definitions?

Lyapunov stable

- basically says that if you give me a neighborhood of the equilibrium, I can find you another neighborhood such that all trajectories starting in the other neighborhood will also stay inside the original neighborhood forever. (Note: the other neighborhood lies inside of the original neighborhood) unstable

- I can give you a neighborhood such that for any other neighborhood you pick, there exists a trajectory starting in the other neighborhood that is gauranteed to leave the original neighborhood at some point in time.





all trajectories starting in $N(x_e;\delta)$ stay in $N(x_e;\epsilon)$

some trajectory starts in $N(x_e;\delta)$ eventually leaves $N(x_e;\epsilon)$

Trajectory:

for any ϵ, never leaves $N(x_e;\epsilon)$ if it starts in $N(x_e;\delta)$ for some δ.

$\delta = \delta(\epsilon, t_0)$

$N(x_e;\delta) \subset N(x_e;\epsilon)$

Trajectory:

there is an ϵ, such that for any δ there is a trajectory starting in $N(x_e;\delta)$ that will eventually leave $N(x_e;\epsilon)$.

(holds trivially if $\delta > \epsilon$)

- for linear systems instability implies blowup, but this is not necessarily so for nonlinear systems.

- once we get the next set of definitions out of the way, we'll contemplate some dynamical systems and their stability properties to get a better feel for things.

Definition. The equilibrium state x_e is <u>uniformly stable</u> (u.s., US) if for arbitrary t_0 and $\epsilon > 0$, $\exists\, \delta(\epsilon): \|x_0 - x_e\| < \delta$ implies that $\|x(t; x_0, t_0) - x_e\| < \epsilon \quad \forall\, t \geq t_0$.

- basically, the δ-neighborhood is independent of the initial time [redacted].

Definition. The equilibrium state x_e is <u>asymptotically stable</u> (a.s., AS) if it is stable and $\exists\, \delta(t_0): \|x_0 - x_e\| < \delta(t_0)$ implies that $\lim_{t \to \infty} \|x(t; x_0, t_0) - x_e\| = 0$.

- doesn't say what happens on the way, just that trajectories starting out close enough end up approaching the equilibrium.

Definition. The equilibrium state x_e is <u>marginally stable</u> if it is stable but not asymptotically stable.

Definition. The equilibrium state x_e is <u>uniformly asymptotically stable</u> (u.a.s., UAS) if it is uniformly stable and for every $\epsilon > 0$ and $t_0 \in \mathbb{R}^+$, $\exists\, \delta_0 > 0$ and a $T(\epsilon) > 0$: $\|x_0 - x_e\| < \delta_0$ implies that $\|x(t; x_0, t_0) - x_e\| < \epsilon \quad \forall\, t \geq t_0 + T(\epsilon)$.

- note that δ_0 is independent of ϵ & t_0, while T is independent of t_0.

Definition. The equilibrium state x_e is exponentially stable (e.s., ES) if

$\exists\, \alpha > 0 : \forall\, \epsilon > 0,\ \exists\, \delta(\epsilon) > 0 : \|x_0 - x_e\| < \delta(\epsilon)$ implies that

$$\|x(t; x_0, t_0) - x_e\| \le \epsilon e^{-\alpha(t - t_0)} \quad \forall\, t \ge t_0.$$

- so, not only does $\lim_{t \to \infty} \|x(t; x_0, t_0) - x_e\| = 0$, but the rate at which it approaches the equilibrium is exponentially fast.

- global stability - the above definitions are said to hold globally if, in their individual definitions, the δ-neighborhood can be arbitrarily large (e.g., the initial state can be arbitrarily far from the equilibrium state). The we can have one or more of the following:

globally stable

globally asymptotically stable (g.a.s, GAS)

globally exponentially stable (g.e.s, GES)

and variations on the others, even instability.

- up to you to read text for the other definitions.

Examples.

1

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -x_1 \end{cases}$$

isolated equilibrium $x_e = (0,0)^T$

$x(0) = (x_{0,1}\ ,\ x_{0,2})^T$

solutions are of the form: $x_1(t) = R\cos(t+\phi_0)$

$x_2(t) = -R\sin(t+\phi_0)$

where,

$\phi_0(x_0) = \text{constant}$

$R = \|x_0\|_2$

The system is stable if I use $\delta(\epsilon) = \epsilon$, or any $0 < \delta(\epsilon) \le \epsilon$.

Since δ is independent of t_0, it is uniformly stable.

It is not asymptotically stable $\Rightarrow$ it is marginally stable.

2] $\dot{x} = -x^3$ isolated equilibrium at $x_e = 0$.

Solution is: $x(t; x_0, t_0) = \left(\dfrac{x_0^2}{1 + 2x_0^2 (t - t_0)} \right)^{1/2}$

It's absolute value is:

$$|x(t; x_0, t_0)| = +\sqrt{\frac{x_0^2}{1 + 2x_0^2 (t - t_0)}} \leq +\sqrt{x_0^2} = |x_0|$$

$\Rightarrow$

for stability, can use $\delta(\epsilon) = \epsilon$. (gives uniform stability)

Furthermore,

$$\lim_{t \to \infty} x(t; x_0, \lambda_0) = 0$$

(gives asymptotic stability)

Convergence to origin holds for any $x_0 \in \mathbb{R}$. (gives global stability)

$\Rightarrow$

system is globally uniformly asymptotically stable.

* to get uniform asy. stability, need additional test. The test holds for any $\delta_0 > 0$ and $T > 0$ when $\delta_0 \leq \epsilon$.

* the system is <u>not</u> exponentially stable.

$|x(t; x_0, t_0)|$ goes as $\sim \frac{1}{\sqrt{2t}}$ for t large enough.

3 van der Pol oscillator

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -x_1 + (1-x_1^2)x_2$$

$\Rightarrow$ find equilibria

$$0 = x_2$$

$$0 = -x_1 + (1-x_1^2)x_2$$

$\Rightarrow$

$x_2 = 0$ and $-x_1 = 0$

$\Rightarrow$

$x_e = (0,0)$, origin is an isolated equilibrium

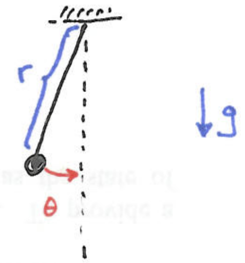
it is, in fact unstable.

- solutions near the origin stay a finite distance away from the origin, but not arbitrarily close to it. They actually diverge from the origin, hence the origin is unstable.

- instability can be shown using $v(x(t)) = \frac{1}{2}x^T(t)x(t)$.

$\rightarrow$ ~~can also use same $v(x(t))$ to show existence of a limit cycle, e.g., a stable trajectory that is a closed curve. All other trajectories are attracted to this limit cycle.~~

4] Pendulum



no damping: $mr^2\ddot{\theta} + mg\sin(\theta) = 0$

$$\Rightarrow \quad x_1 = \theta \;, \quad x_2 = \dot{\theta}$$

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -\frac{g}{r^2}\sin(x_1)$$

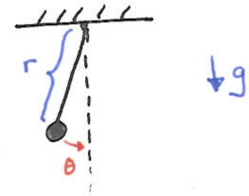
if can swing up, then equilibria are:

$$\left.\begin{aligned} x_2 &= 0 \\ \sin(x_1) &= 0 \end{aligned}\right\} \Rightarrow \begin{aligned} x_1 &= k\pi \\ x_2 &= 0 \end{aligned}$$

for $x_e = ((2k+1)\pi, 0)$, e.g. swing up, the system is not stable. Can you see why?

for $x_e = (2k\pi, 0)$, e.g. swing down, the system is actually marginally stable. Local to the origin, it behaves like Example 1.

4 Pendulum



no damping: $mr^2\ddot{\theta} + mg\sin(\theta) = 0$

$\Rightarrow \; x_1 = \theta \,,\; x_2 = \dot{\theta}$

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -\frac{g}{r^2}\sin(x_1)$$

$\Rightarrow$ equilibria

$$0 = x_2$$
$$0 = -\frac{g}{r^2}\sin(x_1)$$

$\Rightarrow$

$$x_1 = k\pi \;\;\&\;\; x_2 = 0$$

swing up position

$x_e = ((2k+1)\pi, 0) \quad k \in \mathbb{Z}$

unstable

swing down position

$x_e = (2k\pi, 0) \quad k \in \mathbb{Z}$

marginally stable.

with damping: $mr^2\ddot{\theta} + br\dot{\theta} + mg\sin(\theta) = 0$

$\Rightarrow \quad x_1 = \theta \;, \; x_2 = \dot{\theta}$

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -\frac{b}{mr} x_2 - \frac{g}{r^2}\sin(x_1)$$

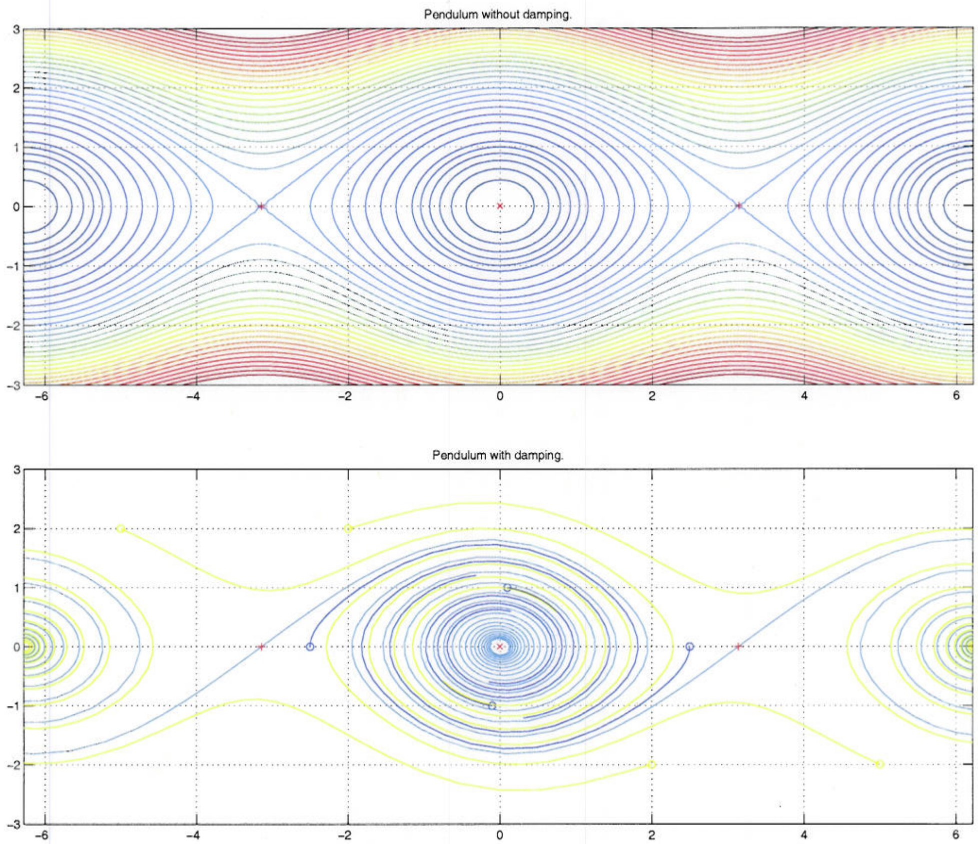
equilibria do not change, but stability type may

for $x_e = ((2k+1)\pi, 0)$ system is still not stable.

for $x_e = (2k\pi, 0)$ the system is asymptotically stable.

(not globally stable due to multiple equilibria)

· notions may change if we operate on S^1 (the circle) as ~~are~~ the space that the ~~dynamics~~ system evolves on



initial conditions of trajectories are at the circle markers.

Color is meant to indicate location of initial condition w/respect to top graph.

Hopefully it makes sense.

Phase plots of pendulum with and without damping.
(BOTTOM) (TOP)

- A phase plot is a plot of $x_1(t)$ vs. $x_2(t)$ in the 2D case.

- The red x marker is the swing down equilibrium.
  red + markers are the swing up equilibria.

- For no damping a "trajectory" connects the swing up $(\pi, 0)$ equilibrium w/the swing up $(-\pi, 0)$ equilibrium. This also holds in the other direction, but on the flip side of the x-axis. These two trajectories form what's called a "~~[illegible]~~ "heteroclinic orbit" and each one takes ∞ time to traverse.

- The heteroclinic orbit is broken and ceases to exist once there is damping.

Before moving on, lets consider other aspects/definitions related to stability.

Definition. The set of all $x_0 \in \mathbb{R}^n$ such that $\lim_{t\to\infty} x(t; x_0, t_0) = x_e$ for some $t_0 \geq 0$ is called the <u>region of attraction</u> of the equilibrium state x_e.

- If the equilibrium x_e satisfies the additional condition for asymptotic stability, then the equilibrium is said to be <u>attractive</u>.

- Thus a property is global if its region of attraction is the entire space $\mathbb{R}^n$.

- AKA <u>domain of attraction</u>

Boundedness:

- The van der Pol oscillator is a good example of a system that is not stable, but exhibits behavior that is not so bad. Solutions are bounded for all time. Sometimes we won't be able to gaurantee stability, but will hopefully be able to demonstrate boundedness.

Definition. A unique, complete solution to the IVP is <u>bounded</u> if there exists a positive constant c, independent of t_0, such that $\|x(t; x_0, t_0)\| < c \quad \forall\, t \geq t_0$, where c may depend on the solution.

→ As was done with stability, it will be useful characterize boundedness with respect to a neighborhood of solutions, as opposed to just one solution.

Definition. Complete, unique solutions to the IVP are

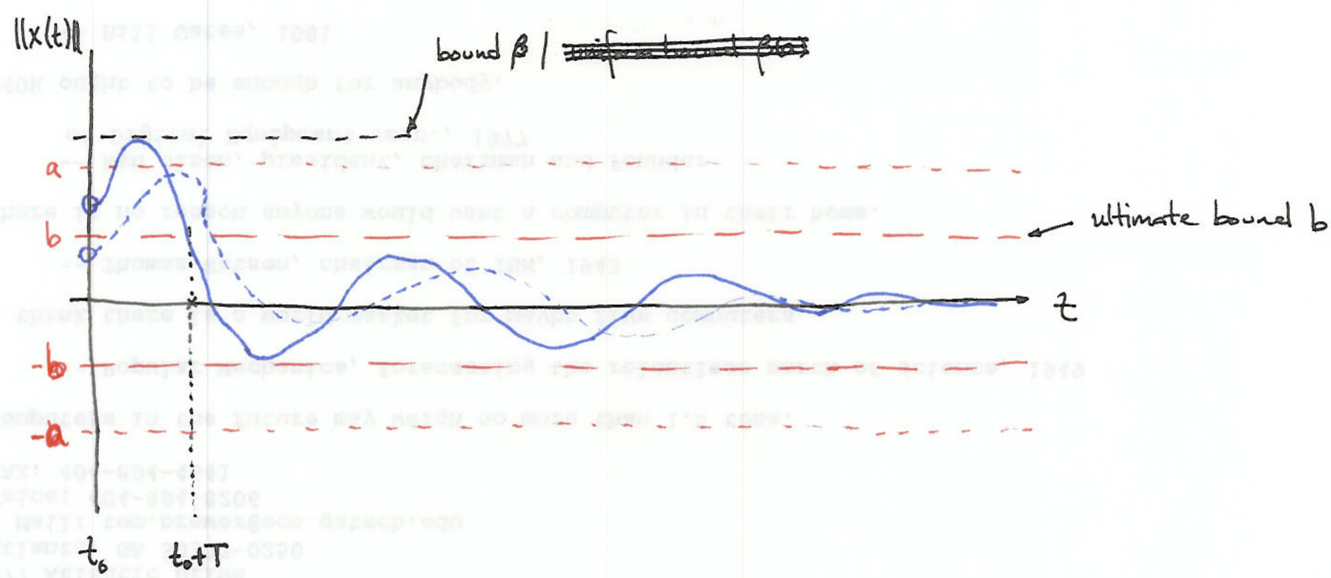
- <u>uniformly bounded</u> if there exists a positive constant c, independent of t_0, such that $\forall a \in (0,c)$, $\exists\, \beta = \beta(a) > 0$, independent of t_0, such that

$$\|x(t_0)\| \leq a \Rightarrow \|x(t)\| \leq \beta \quad \forall\, t \geq t_0 .$$

- <u>globally uniformly bounded</u> if it is uniformly bounded for arbitrary c.
- <u>uniformly ultimately bounded</u> with ultimate bound b, if there exist positive constants b and c, independent of t_0, such that $\forall\, a \in (0,c)$, $\exists\, T = T(a,b)$, independent of t_0, such that

$$\|x(t_0)\| \leq a \Rightarrow \|x(t)\| \leq b \quad \forall\, t \geq t_0 + T .$$

- <u>globally uniformly ultimately bounded</u> if it is uniformly ultimately bounded for arbitrary c.

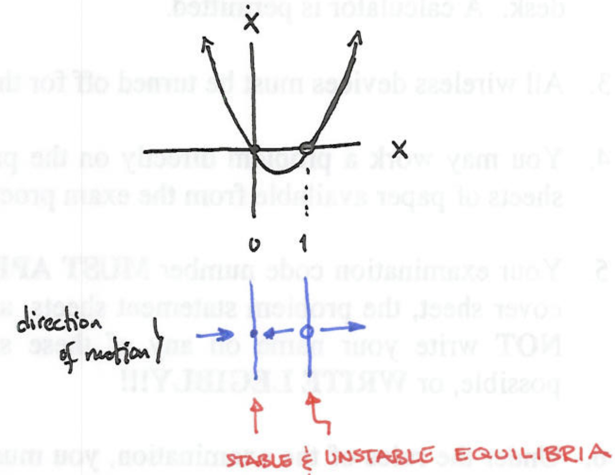


→ Shows two trajectories for some system w/a visual representation of various aspects of boundedness.

* these concepts/definitions extend to functions of space and time.

Examples.

$\boxed{1}$ $\dot{x} = x(x-1)$



· For $x_0 \in (-\infty, 1)$, the solution $x(t; x_0, t_0) \to 0$ as $t \to \infty$.

· If we pick $c = 1$, then

$\forall a \in (0, c)$, choosing $\beta(a) = a$

$\Rightarrow$

$\|x(t; x_0, t_0)\| < a \Rightarrow \|x(t; x_0, t_0)\| \le \beta = a$

$\Rightarrow$

uniformly bounded.

· for $x_0 \in (1, \infty)$, boundedness is not gauranteed.

* boundedness applies to a solution in particular

* uniform boundedness requires a specific neighborhood of solutions to be bounded. Here we use an a-neighborhood of the origin, but neighborhoods around other points can be used if the definitions are properly adjusted or there is a change of variables translating the point in question to the origin.

2. We will work out this solution in two different ways and show two different definitions of boundedness. This is to show that some conclusions are really driven more by the mathematics one uses rather than the practical reality of how the solutions behave.

$$\dot{x} = -x + \delta \sin(t), \qquad x(t_0) = a > \delta > 0$$

solution is

$$x(t) = e^{-(t-t_0)} a + \int_{t_0}^{t} e^{-(t-\tau)} \delta \sin(\tau) d\tau$$

$$= e^{-(t-t_0)} a + \delta \int_{t_0}^{t} e^{-(t-\tau)} \sin(\tau) d\tau$$

$\Rightarrow$

$$|x(t)| \leq \left| e^{-(t-t_0)} a \right| + \left| \delta \int_{t_0}^{t} e^{-(t-\tau)} \sin(\tau) d\tau \right|$$

$\Rightarrow$

$$|x(t)| \leq e^{-(t-t_0)} a + \delta \int_{t_0}^{t} e^{-(t-\tau)} d\tau$$

$$\leq e^{-(t-t_0)} a + \delta \left(1 - e^{-(t-t_0)}\right)$$

$$\leq (a-\delta) e^{-(t-t_0)} + \delta$$

$\Rightarrow$

$$|x(t)| \leq a$$

$\hookrightarrow$ one can conclude uniformly bounded.

noting that the system has exponentially decaying dynamics, one should be able to get a better bound than the initial state.

let $b = (a-\delta) e^{-(t-t_0)} + \delta$

$\Rightarrow$

$$\frac{(b-\delta)}{(a-\delta)} = e^{-(t-t_0)}$$

$\Rightarrow$

$$t - t_0 = -\ln\left(\frac{b-\delta}{a-\delta}\right)$$ flip to change sign of ln.

$\Rightarrow$

$$t = t_0 + \ln\left(\frac{a-\delta}{b-\delta}\right)$$

and we have just shown that

$$|x(t)| \leq b \quad \forall \ t \geq t_0 + T \qquad \text{where } T = T(a,b) = \ln\left(\frac{a-\delta}{b-\delta}\right)$$

where $\delta < b < a$.

So for loose bounds, we can go for uniform boundedness.

For tighter bounds on solutions, we went for uniform ultimate boundedness.

The second solution strategy utilizes $v(x(t)) = \frac{1}{2}x^2(t)$.

$$\frac{d}{dt} v(x(t)) = x(t)\dot{x}(t) = -x^2 + \delta x \sin(t) \leq -x^2 + \delta|x| = -|x|(|x|-\delta)$$

$\Rightarrow$

$$\dot{v}(x(t)) = -|x|(|x|-\delta) < 0 \qquad \forall \ |x| > \delta.$$

Solutions starting outside of $\overline{N}(\delta)$, the closed δ-neighborhood of the origin, enter $\overline{N}(\delta)$ within a finite time interval and remain there.

Formally, if $\alpha > \frac{1}{2}\delta^2$, then all solutions in the set

$$\Omega_\alpha = \{ x \mid v(x) \leq \alpha \}$$

will remain there for all time since $\dot{v} < 0$ on $\partial\Omega_\alpha$ (the boundary of Ω_α). Note that $\overline{N}(\delta) \subset \Omega_c$.

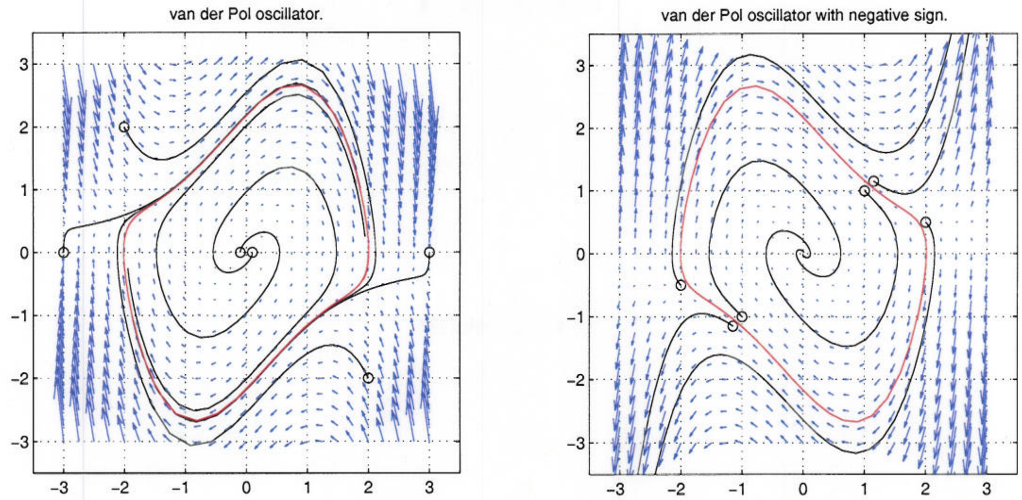
$\Rightarrow$

uniformly bounded.

for $a \in (0,c)$, choosing $\alpha = v(a)$ allows one to show that $\beta(a) = a$ works.

The c to choose is $c \in (\delta, \infty)$; we just need one such c.

3] vanderPol oscillator vs. a variation thereof.



- circles denote initial condition of trajectory.
- arrows are the tangent vectors
-

(a) unstable origin

stable orbit (in red)

(b) stable origin

unstable orbit (in red)

Phase plots of van der Pol osscillator and modified version.

(a) $\dot{x}_1 = x_2$

$\dot{x}_2 = -x_1 + (1 - x_1^2)\, x_2$

vs.

(b) $\dot{x}_1 = x_2$

$\dot{x}_2 = -x_1 + (x_1^2 - 1)\, x_2$

although version (a) has an unstable origin, it does have an orbit around the origin that is stable. Trajectories inside the orbit spiral out to the orbit, while trajectories starting outside of the orbit spiral in to the orbit.

$\Rightarrow$

the system is globally uniformly bounded.

thus, although the origin is unstable, the system trajectories do not blow-up no matter where we start from.

version (b) on the other hand has a stable origin and an unstable orbit. ~~By definition of uniform boundedness~~, We see that the origin has neighborhoods around it that lead one to conclude uniform boundedness. It is not global because outside of the unstable orbit trajectories diverge to ∞.

* if a solution is uniformly bounded, then it is uniformly ultimately bounded. In fact, the opposite also holds: uniform ultimate boundedness $\Rightarrow$ uniform boundedness. Thus, they are equivalent notions. In practice, one form may be easier to prove than another.

Ultimate boundedness vs. Lyapunov stability

* Stability is necessarily defined with respect to an equilibrium (or a limit set), boundedness is not.

* Stability implies I can stay arbitrarily close to an equilibrium point by starting even closer to it. This is too strong of a condition for systems experiencing unknown disturbances.

* For ultimate boundedness, the bound β (equivalent to ϵ in Lyapunov stability) cannot be made arbitrarily small by starting closer to the equilibrium/origin. In practical systems, the bound β depends on the disturbances and system uncertainties.

↳ for an example of this think of the $\delta \sin(t)$ in

$$\dot{x} = -x + \delta \sin(t)$$

as a periodic disturbance. Solution bounds depended on δ, as we saw.