Properties of System Trajectories

- thus far, we've discussed when a solution will exist and what's required for uniqueness / completeness.

- suppose we want to know more? more as in trajectory behavior?

  - how does the trajectory vary as components of the IVP vary?

  - if system dynamics depend on parameters, what happens when they vary?

* The focus of what's next is on analytical properties of system trajectories.

but first, an important Lemma!

Lemma (Gronwall-Bellman Inequality)

Let $\phi: [t_0, t_1] \to \mathbb{R}$ be cts and $\psi: [t_0, t_1] \to \mathbb{R}$ be cts and non-negative.
If a cts function $y: [t_0, t_1] \to \mathbb{R}$ satisfies

$$y(t) \leq \phi(t) + \int_{t_0}^{t} \psi(\tau)\, y(\tau)\, d\tau \qquad \forall\, t \in [t_0, t_1]$$

then

$$y(t) \leq \phi(t) + \int_{t_0}^{t} \exp\left[\int_{\tau}^{t} \psi(\sigma)\, d\sigma\right] \phi(\tau)\, \psi(\tau)\, d\tau \qquad \forall\, t \in [t_0, t_1].$$

If $\phi(t) = \phi_0$ constant, then

$$y(t) \leq \phi_0 \exp\left[\int_{t_0}^{t} \psi(\tau)\, d\tau\right]$$

and if it also holds that $\psi(t) = \psi_0$ constant, then

$$y(t) \leq \phi_0 \exp\left[\psi_0 (t - t_0)\right].$$

proof.

we have $$y(t) \le \phi(t) + \int_{t_0}^{t} \psi(\tau)\, y(\tau)\, d\tau \qquad (*)$$

let $$z(t) \equiv \int_{t_0}^{t} \psi(\tau)\, y(\tau)\, d\tau$$

and $$v(t) \equiv z(t) + \phi(t) - y(t) \geq 0$$

holds by $(*)$

differentiating $z(t)$ w/respect to t,

$$\dot{z}(t) = \psi(t)\, y(t) = \psi(t)\left(z(t) + \phi(t) - v(t) \right)$$

$$= \psi(t)\, z(t) + \psi(t)\left[\phi(t) - v(t)\right]$$

$\Rightarrow$ integrate

$$z(t) = \Phi_{t_0,t} \cdot z_0 + \int_{t_0}^{t} \Phi_{\tau,t} \cdot \psi(\tau)\left[\phi(\tau) - v(\tau)\right] d\tau,$$

where

$$\Phi_{t_0,t} = \exp\left[\int_{t_0}^{t} \psi(\tau)\, d\tau\right]$$

$\Rightarrow$ but, by definition of $z(t)$, $z(t_0) = 0$, so $z_0 = 0$

$$z(t) = \int_{t_0}^{t} \Phi_{\tau,t} \cdot \psi(\tau)\left[\phi(\tau) - v(\tau)\right] d\tau$$

but since $v(\tau) \geq 0$ and $\psi(\tau)$ is non-negative, we know that

$$z(t) = \int_{t_0}^{t} \Phi_{\tau,t} \cdot \psi(\tau)\left[\phi(\tau) - v(\tau)\right] d\tau \le \int_{t_0}^{t} \Phi_{\tau,t} \cdot \psi(\tau)\, \phi(\tau)\, d\tau$$

this part removed, since it evaluates to a positive number and therefore subtracts from the first term due to the minus sign.

$\Rightarrow$

$$z(t) \le \int_{t_0}^{t} \exp\left[\int_{\tau}^{t} \psi(\sigma)\, d\sigma\right] \cdot \psi(\tau)\, \phi(\tau)\, d\tau$$

$\Rightarrow$ Now,

$$y(t) \leq \phi(t) + z(t)$$

$\Rightarrow$

$$y(t) \leq \phi(t) + \int_{t_0}^{t} \exp\left[\int_{\tau}^{t} \psi(\sigma)d\sigma\right]\psi(\tau)\phi(\tau)d\tau \quad .$$

Now, if $\phi(t) = \phi_0$, then substitution and integration leads to

$$y(t) \leq \phi_0 \exp\left[\int_{t_0}^{t} \psi(\sigma)d\sigma\right]$$

and if $\psi(t) = \psi_0$ also, then

$$y(t) \leq \phi_0 \exp\left[\psi_0(t-t_0)\right].$$

- The Gronwall-Bellman inequality is immensely useful in providing explicit bounds for implicit inequalities.
  (implicitly bounded quantities)

- Comes up very often in the study of dynamical systems.

Theorem (Continuous Dependence of System Trajectories on Time).

Let $f(x,t)$ be piecewise cts in t and uniformly Lipschitz cts in x and uniformly in t on $D \times [t_0, t_1]$. Then the solution $x(t; x_0, t_0)$ to the IVP is cts with respect to t_0.

proof.

Suppose we had two IVP's, one with initial time t_0 and the other with $t_0' > t_0$, then we can get continuity by showing that

$$\underbrace{\| x(t; x_0, t_0)}_{1^{st}\text{ solution}} - \underbrace{x(t; x_0, t_0') \|}_{2^{nd}\text{ solution}} \le M (t_0' - t_0) \exp(\underbrace{L}_{\text{going to be Lipschitz const.}}(t - t_0'))$$

for some $M > 0$, where both solutions exist on the interval $[t_0, t_1]$.

To see this, examine the two solutions for $t \in [t_0, t_1]$

$$x(t; x_0, t_0) = x_0 + \int_{t_0}^{t} f(x(\tau; x_0, t_0), \tau)\, d\tau$$

$$x(t; x_0, t_0') = x_0 + \int_{t_0}^{t} f(x(\tau; x_0, t_0'), \tau)\, d\tau$$

$\Rightarrow$

$$\| x(t; x_0, t_0) - x(t; x_0, t_0') \| = \left\| \int_{t_0}^{t_0'} f(x(\tau; x_0, t_0), \tau)\, d\tau + \int_{t_0'}^{t} [f(x(\tau; x_0, t_0), \tau) - f(x(\tau; x_0, t_0'), \tau)]\, d\tau \right\|$$

$\Rightarrow$ TRIANGLE INEQUALITY

$$\le \left\| \int_{t_0}^{t_0'} f(x(\tau; x_0, t_0), \tau)\, d\tau \right\| + \left\| \int_{t_0'}^{t} [f(x(\tau; x_0, t_0), \tau) - f(x(\tau; x_0, t_0'), \tau)]\, d\tau \right\|$$

$\Rightarrow$ $\| \int f \| \le \int \| f \|$

$$\le \int_{t_0}^{t_0'} \| f(x(\tau; x_0, t_0), \tau) \|\, d\tau + \int_{t_0'}^{t} \| f(x(\tau; x_0, t_0), \tau) - f(x(\tau; x_0, t_0'), \tau) \|\, d\tau$$

now, f has the properties that allow one to assert that $\exists\, M > 0$ such that

$$\| f(x(t;x_0,t_0),t) \| \leq M \quad \text{on the } \underline{\text{finite}} \text{ time interval } [t_0,t_1].$$

$\Rightarrow$

$$\|x(t;x_0,t_0) - x(t;x_0,t_0')\| \leq \int_{t_0}^{t_0'} M\,d\tau + \int_{t_0'}^{t} \| f(x(\tau;x_0,t_0),\tau) - f(x(\tau;x_0,t_0'),\tau)\|\,d\tau$$

furthermore, since f is uniformly Lipschitz cts in x uniformly in t, we can assert that $\exists\, L > 0$ such that

$$\| f(x(\tau;x_0,t_0),\tau) - f(x(\tau;x_0,t_0'),\tau)\| \leq L \,\| x(\tau;x_0,t_0) - x(\tau;x_0,t_0')\|$$

$\Rightarrow$

$$\| x(t;x_0,t_0) - x(t;x_0,t_0')\| \leq \int_{t_0}^{t_0'} M\,d\tau + \int_{t_0'}^{t} L\,\| x(\tau;x_0,t_0) - x(\tau;x_0,t_0')\|\,d\tau$$

$$\underbrace{\|x(t;x_0,t_0) - x(t;x_0,t_0')\|}_{y(t)} \underset{\leq}{\leq} \underbrace{M(t_0'-t_0)}_{\phi_0} + \int_{t_0'}^{t} \underbrace{L}_{\psi_0} \underbrace{\|x(\tau;x_0,t_0) - x(\tau;x_0,t_0')\|}_{y(\tau)}\,d\tau$$

$$y(t) \leq \phi_0 + \int_{t_0'}^{t} \psi_0 \cdot y(\tau) \cdot d\tau$$

$\Rightarrow$ invoke Gronwall-Bellman Lemma

$$\|x(t;x_0,t_0) - x(t;x_0,t_0')\| \leq \phi_0 \exp[\psi_0(t-t_0')] = M(t_0'-t_0)\exp[L(t-t_0')],$$

To get continuity, notice that on the finite interval $[t_0,t_1]$, for all $\epsilon > 0$, if we define $\delta(\epsilon) \equiv \dfrac{\epsilon}{M\exp[L(t-t_0')]}$, then

$$|t_0' - t_0| \leq \delta \quad \text{implies} \quad \|x(t;x_0,t_0) - x(t;x_0,t_0')\| \leq \epsilon.$$

Theorem (Bounded Perturbations of the State Dynamics)

Let $f(x,t)$ be piecewise cts in t, uniformly Lipschitz cts in x, uniformly in t, on the domain $D \times [t_0, t_1]$ where D is an open, connected set.

Let $y(t)$ and $z(t)$ be solutions to the IVPs

$$\dot{y}(t) = f(y,t) \quad , \quad y(t_0) = y_0 \quad ,$$

and

$$\dot{z}(t) = f(y,t) + g(y,t) \quad , \quad z(t_0) = z_0 \quad ,$$

which lie in the set D over the time interval $[t_0, t_1]$. Suppose that g is uniformly bounded $\underset{\text{in } D\times[t_0,t_1]}{\wedge}$, e.g., $\exists \mu > 0$:

$$\| g(x,t) \| \leq \mu \qquad \forall (x,t) \in D \times [t_0, t_1].$$

Then,

$$\|y(t) - z(t)\| \leq \|y_0 - z_0\| \exp[L(t-t_0)] + \frac{\mu}{L}\left(\exp[L(t-t_0)] - 1\right) \quad \forall t \in [t_0, t_1].$$

proof.

$$y(t) = y_0 + \int_{t_0}^{t} f(y(\tau), \tau)\, d\tau$$

$$z(t) = z_0 + \int_{t_0}^{t} [f(z(\tau),\tau) + g(z(\tau),\tau)]\, d\tau$$

$\Rightarrow$ take norm of difference & use triangle inequality (twice)

$$\|y(t) - z(t)\| \leq \|y_0 - z_0\| + \left\| \int_{t_0}^{t} [f(y(\tau),\tau) - f(z(\tau),\tau)]\, d\tau \right\| + \left\| \int_{t_0}^{t} g(z(\tau),\tau)\, d\tau \right\|$$

$\Rightarrow$ $\|\int f\| \leq \int \|f\|$

$$\leq \|y_0 - z_0\| + \int_{t_0}^{t} \|f(y(\tau),\tau) - f(z(\tau),\tau)\|\, d\tau + \int_{t_0}^{t} \|g(z(\tau),\tau)\|\, d\tau$$

$\Rightarrow$ g bounded

$$\leq \|y_0 - z_0\| + \int_{t_0}^{t} \mu\, d\tau + \int_{t_0}^{t} L\, \|y(\tau) - z(\tau)\|\, d\tau$$

Lipschitz constant due to uniform Lipschitz continuity.

So, we are at

$$\|y(t)-z(t)\| \le \|y_0 - z_0\| + \mu(t-t_0) + \int_{t_0}^{t} L\,\|y(\tau)-z(\tau)\|\,d\tau$$

$$\underbrace{\|y(t)-z(t)\|}_{\xi(t)} \underbrace{\le}_{\le} \underbrace{\gamma + \mu(t-t_0)}_{\phi(t)} + \underbrace{\int_{t_0}^{t}}_{\int_{t_0}^{t}} \underbrace{L}_{L}\, \underbrace{\|y(\tau)-z(\tau)\|}_{\xi(\tau)}\,\underbrace{d\tau}_{d\tau}$$

$\Rightarrow$ Gronwall-Bellman inequality

$$\|y(t)-z(t)\| \le \gamma + \mu(t-t_0) + \int_{t_0}^{t} \exp[L(t-\tau)]\cdot(\gamma + \mu(\tau - t_0))\cdot L\,d\tau$$

$$\le \gamma + \mu(t-t_0) + \underbrace{\int_{t_0}^{t} \exp[L(t-\tau)]\,\gamma L\,d\tau}_{\text{integrate}} + \underbrace{\int_{t_0}^{t} \underbrace{\exp[L(t-\tau)]}_{du}\cdot \underbrace{\mu(\tau-t_0)}_{v} L\,d\tau}_{\text{integrate by parts}}$$

$\Rightarrow$

$$\le \gamma \exp[L(t-t_0)] + \frac{\mu}{L}\left(\exp[L(t-t_0)] - 1\right)$$

$\Rightarrow$

$$\|y(t)-z(t)\| \le \|y_0 - z_0\| \exp[L(t-t_0)] + \frac{\mu}{L}\left(\exp[L(t-t_0)] - 1\right)$$

□

MEANING: • a bounded perturbation gives rise to exponential divergence from the unperturbed system in the worst case.

(this is preferred to blow-up in finite time)

• therefore for finite time, the two solutions are a finite distance apart from each other. This could be useful.

Theorem (Continuous Dependence on Initial Conditions and Parameters)

Let $f(x,t;\lambda)$ be cts in (x,t,λ), Lipschitz cts in x (uniformly in λ & t) on the domain $D \times [t_0,t_1] \times \Lambda$, where D is open and connected and $\Lambda \equiv N(\lambda_0; c)$, for $c>0$, is a neighborhood of λ_0. Further, let $x(t;\lambda_0)$ be a solution to the IVP

$$\dot{x} = f(x,t;\lambda_0) \quad , \quad x(t_0;\lambda_0) = x_0 \in D$$

defined over the interval $[t_0,t_1]$ and lying in D for all $t \in [t_0,t_1]$.

Then $\forall \epsilon > 0, \exists \delta > 0$: if $\|z_0 - y_0\| < \delta$ and $\|\lambda - \lambda_0\| < \delta$, there is a unique solution $z(t;\lambda)$ of the IVP

$$\dot{z} = f(z,t;\lambda) \quad , \quad z(t_0;\lambda) = z_0 \in D$$

defined on $[t_0,t_1]$ that satisfies

$$\| z(t,\lambda) - x(t,\lambda_0) \| < \epsilon \qquad \forall\, t \in [t_0,t_1].$$

for proof see Hovakimyan lecture notes

or Khalil textbook

or most any other text covering dynamical systems theory.

Bounds on State Trajectories

- proof of continuity theorems for system trajectories utilized boundedness properties of the solution on finite time intervals.

- can we still find bounds if the time interval can be arbitrary?

  ↳ yes, with a diff. eqn. version of the bounding theorems for series, integrals, etc.

  idea is to use a known quantity to limit/bound an unknown quantity.

Lemma (Comparison Lemma). Consider the scalar differential equation

$$\dot{u} = g(u,t), \quad u(t_0) = u_0$$

where $g(u,t)$ is cts in t and Lipschitz cts in u on the domain $D \subset \mathbb{R}$ uniformly in $t \geq 0$. Let $[t_0, T)$ be the maximal interval of existence of the solution $u(t)$, and suppose that $u(t) \in D \;\; \forall t \in [t_0, T)$. Let $v(t)$ be a cts function whose upper right-hand derivative $D^+ v(t)$ satisfies the differential inequality

$$D^+ v(t) \leq g(v,t), \qquad v(t_0) \leq u_0$$

with $v(t) \in D$, $\forall t \in [t_0, T)$. Then $v(t) \leq u(t)$.

Example.

$\boxed{1}$ $\dot{x} = -(1+x^2)x \qquad x(0) = a$

$\rightarrow$ what's going on w/solution $x(t)$?

Well, suppose that $v(t) = x^2(t)$

$\Rightarrow$

$$\dot{v}(t) = 2x(t)\dot{x}(t) = -2x^2(t) - 2x^4(t) \leq -2x^2(t)$$

$\Rightarrow$

$$\dot{v}(t) \leq -2v(t)$$

pick $g(u,t) = -2u(t) \Rightarrow \dot{u}(t) = -2u(t)$

$\Rightarrow$

for $u(0) = x^2(0) = a^2$, $u(t) = a^2 e^{-2t}$

$\Rightarrow$

$$v(t) \leq u(t) \leq a^2 e^{-2t}$$

$\Rightarrow$

$$v(t) \leq a^2 e^{-2t}$$

How does this help?

Well, since $v(t) = x^2(t)$

$\Rightarrow$

$$x^2(t) \leq a^2 e^{-2t}$$

$\Rightarrow$

$$|x(t)| \leq |a| e^{-t}$$

$x(t)$ is bounded by $|a|$.

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$$\dot{x} = -x + \frac{\sin(t)}{1+x^2} \qquad x(0) = 2$$

Again, let $v(t) = \frac{1}{2}x^2(t)$

$\Rightarrow$

$$\dot{v}(t) = x(t)\dot{x}(t) = -x^2 + \frac{x\sin(t)}{1+x^2} \leq -x^2 + 1$$

$\Rightarrow$

$$\dot{v}(t) \leq -2v(t) + 1$$

Since $v(0) = \frac{1}{2}x^2(0) = 2$, we setup the companion IVP to be

$$\dot{u} = -2u + 1 \qquad u(0) = 2$$

$\Rightarrow$ integrate to get solution

$$u(t) = \frac{1 + 3e^{-2t}}{2}$$

$\Rightarrow$

$$v(t) \leq u(t) = \frac{1+3e^{-2t}}{2}$$

$\Rightarrow$

$$|x(t)| = \sqrt{2v(t)} \leq \sqrt{1+3e^{-2t}}$$

- although we have no idea what happens to $x(t)$, we know that its absolute value is bounded.

[redacted]