Adaptive Backstepping

- The adaptive controllers examined cases where the uncertainty was matched. What if we don't match?

  ↳ can try backstepping.

Example:
$$\dot{x}_1 = x_2 + \alpha \varphi(x_1)$$
$$\dot{x}_2 = u$$

does not fit form of: $\dot{x} = Ax + b(u + \alpha \Phi(x))$

since $b = \begin{Bmatrix} 0 \\ 1 \end{Bmatrix}$ and nonlinear portion $\alpha \begin{Bmatrix} \varphi(x_1) \\ 0 \end{Bmatrix}$ does not lie in the span of b.

Aerospace Example:
$$\dot{\alpha} = -L_\alpha(\alpha) + q$$
$$\dot{q} = M_0(\alpha, q) + u$$

α - angle of attack

q - pitch rate

L_α - unknown lift coefficient

M_0 - unknown pitch moment

- slightly more complicated than the first example.
- let's baby step our way to being able to even consider this case.

Claim was that adaptive backstepping will do the trick. OK, but what does that mean?

- we'll get to explaining "adaptive backstepping" with increasingly representable ~~problem~~ examples.

Integrator Backstepping

- first we consider backstepping w/out the adaptive part.

Start with the following (nonlinear) control system:

$$\dot{x} = f(x) + g(x)\xi$$
$$\dot{\xi} = u$$

Complete state is $z = (x, \xi)^T \in \mathbb{R}^2$
Control input is $u(t) \in \mathbb{R}$
Constitutive functions are $f, g \in C^\infty(D; \mathbb{R})$ where $0 \in D \subset \mathbb{R}^2$ and $f(0) = 0$.

necessary for origin to be equilibrium.

GOAL: design state feedback control law for asymptotically stabilizing the origin

- Idea is to treat ξ as an input since we can directly control it.
  But, the fact that we achieve this through $\dot{\xi}$ means we need to be careful about the choice of u, since errors in ξ will integrate up if we are not.

The first thing to do is to redefine the dynamics (change of coordinates)

$$\dot{x} = f(x) + g(x)\,\xi_c(x) + g(x)\,\Delta\xi$$

$$\Delta\dot{\xi} = v$$

where $\xi_c(x)$ is the desired control law for stabilizing x,

and $\Delta\xi = \xi(t) - \xi_c(t)$ plus $v = u - \dot{\xi}_c$

$$\dot{\xi}_c = \frac{\partial \xi_c}{\partial x} \cdot \dot{x} = \frac{\partial \xi_c}{\partial x}\left[f(x) + g(x)\xi\right]$$

in ideal case:

$$\xi_c = \frac{1}{g(x)}\left(-k_1 x - f(x)\right)$$

$\Rightarrow$

$$\dot{x} = -k_1 x + g(x)\,\Delta\xi$$

Note: $g(x)$ cannot vanish, otherwise there is a loss of control.

With above definition, substitution of $\xi_c(x)$ into system gives

$$\dot{x} = -k_1 x + g(x)\,\Delta\xi$$

$$\dot{\xi} = v$$

What should v be to result in stability?

If $v = -k_2\Delta\xi - g(x)x$, then

$$\begin{Bmatrix} \dot{x} \\ \Delta\dot{\xi} \end{Bmatrix} = \begin{bmatrix} -k_1 & g(x) \\ -g(x) & -k_2 \end{bmatrix} \begin{Bmatrix} x \\ \Delta\xi \end{Bmatrix}$$

has skew-symmetry

choose

$$V(x, \Delta\xi) = \tfrac{1}{2}x^2 + \tfrac{1}{2}\Delta\xi^2$$

$\Rightarrow$

$$\dot{V} = -k_1 x^2 + g(x)\Delta\xi + \Delta\xi \cdot v$$

$\Rightarrow$ $v = -k_2\Delta\xi - g(x)x$

$$\dot{V} = -k_1 x^2 - k_2\Delta\xi^2$$

$\Rightarrow$

STABILITY!!! (OF x & $\Delta\xi$).

- Saw nonlinear version of this in HW3, prob. 4 I believe.
- Completion of skew-symmetry using v was crucial.

(this was the term leading to integration of control error and we pretty much neutralized it.)

Let's step through this once more to generalize the math, if at all possible.

[1] Stabilize system with pseudo control $\xi_c(t)$,

$$\dot{x} = f(x) + g(x)\,\xi_c(t)$$

and get the associated Lyapunov function $V(x(t))$.

$$\dot{V} = \frac{\partial V}{\partial x}\left[f + g\xi_c\right] \leq -W(x) < 0$$

[2] Consider the complete, augmented system,

$$\begin{Bmatrix} \dot{x} \\ \dot{\Delta\xi} \end{Bmatrix} = \begin{Bmatrix} f(x) + g(x)\xi_c(x) + g(x)\,\Delta\xi \\ v \end{Bmatrix}$$

where $v = u - \dot{\xi}_c$. For the augmented system define the candidate Lyapunov function

$$\mathcal{V} = V + \tfrac{1}{2}\Delta\xi^2$$

[3] Find v such that $\dot{\mathcal{V}} \leq 0$.

This usually means $v = -\frac{\partial V}{\partial x} g(x) - k\,\Delta\xi$

$$\dot{\mathcal{V}} = \dot{V} + \frac{\partial V}{\partial x} g(x)\,\Delta\xi + \Delta\xi \cdot v$$

$$\leq -W + \left(\frac{\partial V}{\partial x} g(x) + v\right)\Delta\xi$$

$$\leq -W(x) - k\,\Delta\xi^2$$

[4] Find complete control law u,

$$u = v + \dot{\xi}_c = -\frac{\partial V}{\partial x} g(x) - k\,\Delta\xi + \frac{\partial \xi_c}{\partial x}\dot{x}$$

$\Rightarrow$

$$u = \frac{\partial \xi_c}{\partial x}\left[f(x) + g(x)\xi\right] - \frac{\partial V}{\partial x} g(x) - k\left[\xi - \xi_c(x)\right]$$

and, we're done! We get an asymptotically stable origin.

If V is radially unbounded, then origin is globally asymptotically stable.

A little bit more work shows that $\dot{x}(t) \to 0$ and $\Delta \xi(t) \to 0$ for $t \to \infty$

implies that $x(t) \to 0$ and $\xi(t) \to 0$ as $t \to \infty$.

Idea: $x(t) \to 0$ usually means $\xi_c(t) \to 0$

so $\Delta \xi(t) = \xi(t) - \xi_c(t) \to 0$ means that $\xi(t) \to 0$.

What about more general systems?

- Well, it's pretty much the same idea.

Generalization:

$$\dot{x}_1 = f_1(x_1) + g_1(x_1)\, x_2$$

$$\dot{x}_2 = f_2(x_1, x_2) + g_2(x_1, x_2)\, u$$

$f_i, g_i \in C^\infty(D; \mathbb{R})$ where D is the domain appropriate to each function.

Also require that $f_1(0) = 0$ and that $g_i \neq 0$ over domain of interest.

[1] Simplify / Linearize control dynamics (x_2)

$$u = \frac{1}{g_2(x_1, x_2)} \left[\hat{u} - f_2(x_1, x_2) \right]$$

$\Rightarrow$

$$\dot{x}_2 \equiv \hat{u}$$

[2] Determine stabilizing control for x_1 dynamics

- this defines $x_{2,c}(x_1)$ along with an associated Lyapunov function $V(x_1)$ such that $\dot{V} \leq -W < 0$.

[3] Consider the augmented system and a candidate Lyapunov function

- candidate is $\mathcal{V} = V + \frac{1}{2}(\Delta x_2)^2$, $\Delta x_2 = x_2 - x_{2,c}$

[4] Get negative definite $\dot{\mathcal{V}}$

- negative definiteness helps defined augmented system control ν, which gives $\hat{u} = \nu + \dot{\xi}_c$.

typically, $\nu = -\frac{\partial V}{\partial x_1} g_1 - k\, \Delta x_2$.

$\boxed{5}$ Complete u is

$$u = \frac{1}{g_2(x_1,x_2)}\left[\underbrace{\frac{\partial x_{2,c}}{\partial x_1}\left(f_1(x_1) + g_1(x_1)x_2\right)}_{\dot{\xi}_c} - \underbrace{\frac{\partial V}{\partial x_1} g_1(x_1) - k(x_2 - x_{2,c})}_{v} - f_2(x_1,x_2)\right]$$

DONE!

This method extends to cascaded nonlinearities.

$$\dot{x}_1 = f_1(x_1) + g_1(x_1)x_2$$
$$\dot{x}_2 = f_2(x_1,x_2) + g_2(x_1,x_2)x_3$$
$$\vdots$$
$$\dot{x}_{n-1} = f_{n-1}(x_1,\ldots,x_{n-1}) + g_{n-1}(x_1,\ldots,x_{n-1})x_n$$
$$\dot{x}_n = f_n(x_1,\ldots,x_n) + g_n(x_1,\ldots,x_n)u$$

- Chained form systems are a good example of this.

Integrator Backstepping + Adaptation:

Baby stepping our way to the full solution, we are going to check out backstepping + adaptation to matched uncertainty.

We will examine the following example

$$\dot{x}_1(t) = \varphi_1(x_1(t)) + x_2(t)$$
$$\dot{x}_2(t) = \alpha\varphi_2(x_1(t)) + u(t)$$

this is our uncertainty. still matched since it lies in the span of the control.

GOAL: Stabilize to origin. For origin to be equilibrium point, need $\varphi_1(0)=0$.

↳ Let's loosely follow the prescribed procedure.

[1] Simplify/Linearize the control dynamics (x_2)

- in this case, we also adapt to the unknown α.

$$u(t) = \hat{u}(t) - \hat{\alpha}(t)\varphi_2(x_1(t))$$

$\Rightarrow$

$$\dot{x}_1(t) = \varphi_1(x_1(t)) + x_2(t)$$
$$\dot{x}_2(t) = \hat{u}(t) - \Delta\alpha(t)\varphi_2(x_1(t))$$

adaptive part is new contribution compared to original method. Let's try to keep adaptive part separate so we can deal with it at the proper time.

$\boxed{2}$ Stabilizing pseudo-control for x_1.

$$\dot{x}_1(t) = \varphi_1(x_1(t)) + x_{2,c}(t)$$

$\Rightarrow$

$$x_{2,c}(t) \equiv -k_1 x_1(t) - \varphi_1(x_1(t))$$

The Lyapunov function $V(x_1) = \frac{1}{2}x_1^2$ shows asymptotic stability.

$\boxed{3}$ Augmented system + adaptation.

$$\dot{x}_1(t) = \varphi_1(x_1(t)) + x_{2,c}(t) + \Delta x_2(t)$$

$$\Delta\dot{x}_2(t) = \hat{\upsilon}(t) - \Delta\alpha(t)\,\varphi_2(x_1(t))$$

trying, still, to keep adaptation separate.

$\Rightarrow$ $\hat{\upsilon} = \hat{u} - \dot{x}_{2,c}$

substitute in $x_{2,c}(t)$

$$\dot{x}_1(t) = -k_1 x_1(t) + \Delta x_2(t)$$

$$\Delta\dot{x}_2(t) = \hat{\upsilon}(t) - \Delta\alpha(t)\,\varphi_2(x_1(t))$$

candidate Lyapunov function : $\mathcal{V} = V + \frac{1}{2}\Delta x_2^2(t) + \frac{1}{2}\gamma\,\Delta\alpha^2(t)$

4 Figure out how to get $\dot{V} \le 0$.

$$\dot{V} = x_1(t)\,\dot{x}_1(t) + \Delta x_2(t)\,\Delta\dot{x}_2(t) + \gamma\,\Delta\alpha(t)\,\Delta\dot{\alpha}(t)$$

$$= x_1(t)\left[-k_1 x_1(t) + \Delta x_2(t)\right] + \Delta x_2(t)\left[\hat{u}(t) - \Delta\alpha(t)\varphi_2(x_1(t))\right] + \gamma\,\Delta\alpha(t)\,\Delta\dot{\alpha}(t)$$

$$= -k_1 x_1^2(t) + \Delta x_2(t)\big[\underbrace{x_1(t) + \hat{u}(t)}\big] + \Delta\alpha(t)\big[\underbrace{-\Delta x_2(t)\varphi_2(x_1(t)) + \gamma\,\Delta\dot{\alpha}(t)}\big]$$

want to vanish or be neg. definite

$\Rightarrow$

choose
$$\begin{cases} \hat{u}(t) = x_1(t) - k_2\,\Delta x_2(t) \\ \Delta\dot{\alpha}(t) = \dot{\hat{\alpha}}(t) = \gamma\,\Delta x_2(t)\,\varphi_2(x_1(t)) \end{cases}$$

and

$$\dot{V} = -k_1 x_1^2(t) - k_2\,\Delta x_2^2(t)$$

$\Rightarrow$

$$x_1(t) \to 0 \quad \& \quad \Delta x_2(t) \to 0 \quad \text{as } t \to \infty$$

but $\Delta x_2(t) = x_2(t) - x_{2,c}(t)$, so what happens now?

$\longrightarrow$ well, as before we know that

$$x_{2,c}(t) = -k_1 x_1(t) - \varphi_1(x_1(t))$$

this is another place where $\varphi_1(0) = 0$ is important.

$\Rightarrow$

$$x_{2,c}(t) \to 0 \quad \text{as } t \to \infty$$

$\Rightarrow$

$$x_1(t) \to 0 \quad \& \quad x_2(t) \to 0 \quad \text{as } t \to \infty$$

and $\Delta\alpha(t)$ is bounded for all time $\Rightarrow$ $\hat{\alpha}(t)$ is bounded.

$\boxed{5}$ Put it all together to get u.

$$\hat{u} = \hat{\upsilon} + \dot{x}_{2,c}(t)$$

$$= x_1(t) - k_2 \Delta x_2(t) + \frac{\partial x_{2,c}}{\partial x_1} \dot{x}_1(t)$$

$$= x_1(t) - k_2 \Delta x_2(t) - (k_1 + \partial\varphi_1/\partial x_1)(-k_1 x_1(t) + \Delta x_2(t))$$

$$\Rightarrow u = \hat{u} - \hat{\alpha}\varphi_2(x_1(t))$$

$$\begin{cases} u(t) = x_1(t) - k_2 \Delta x_2(t) - (k_1 + \partial\varphi_1/\partial x_1)(-k_1 x_1(t) + \Delta x_2(t)) - \hat{\alpha}(t)\varphi_2(x_1(t)) \\ \dot{\hat{\alpha}}(t) = \gamma \Delta x_2(t) \varphi_2(x_1(t)) \end{cases}$$

OR

$$\begin{cases} u(t) = x_1(t) - k_2 \Delta x_2(t) - (k_1 + \partial\varphi_1/\partial x_1)(\varphi_1(x_1(t)) + x_2(t)) - \hat{\alpha}(t)\varphi_2(x_1(t)) \\ \dot{\hat{\alpha}}(t) = \gamma \Delta x_2(t) \varphi_2(x_1(t)) \end{cases}$$

Adaptive Backstepping:

Finally, we get to our real problem:

- unmatched, unknown parameters.

Again, we will consider the simpler goal of stabilization and **NOT** tracking.

$$\dot{x}_1(t) = x_2(t) + \alpha\varphi(x_1(t))$$

$$\dot{x}_2(t) = u$$

now uncertainty is definitely unmatched.

As before, let's follow the procedure making adjustments as necessary.

$\boxed{1}$ Stabilize x_1 dynamics (w/ adaptation)

$$\dot{x}_1(t) = x_{2,c}(t) + \alpha\varphi(x_1(t))$$

$\Rightarrow$ pseudo control has adaptive part now.

$$x_{2,c}(t) = -k_1 x_1(t) - \hat{\alpha}(t)\varphi(x_1(t))$$

$\Rightarrow$

$$\dot{x}_1(t) = -k_1 x_1(t) - \Delta\alpha(t)\varphi(x_1(t))$$

← if the world were perfect this would be it, but it's not so...

- due to adaptive part, we will forgo defining a Lyapunov function here.

$\boxed{3}$ get $\dot{V}$ negative semi definite.

Taking time derivative gives

$$\dot{V} = x_1(t)\dot{x}_1(t) + \Delta x_2(t)\Delta\dot{x}_2(t) + \gamma^{-1}\Delta\alpha(t)\Delta\dot{\alpha}(t)$$

$$= x_1(t)\left[-k_1 x_1(t) + \Delta x_2(t) - \Delta\alpha(t)\varphi(x_1(t))\right]$$

$$+ \Delta x_2(t)\left[u - \frac{\partial x_{2,c}}{\partial x_1}\dot{x}_1(t) + \varphi(x_1(t))\dot{\hat{\alpha}}(t)\right]$$

$$+ \gamma^{-1}\Delta\alpha(t)\Delta\dot{\alpha}(t)$$

$\Rightarrow$ expand $\dot{x}_1(t)$

$$\dot{V} = x_1(t)\left[-k_1 x_1(t) + \Delta x_2(t) - \Delta\alpha(t)\varphi(x_1(t))\right]$$

$$+ \Delta x_2(t)\left[u - \partial x_{2,c}/\partial x_1\left(-k_1 x_1(t) + \Delta x_2(t) - \Delta\alpha(t)\varphi(x_1(t))\right) + \varphi(x_1(t))\dot{\hat{\alpha}}(t)\right]$$

$$+ \gamma^{-1}\Delta\alpha(t)\Delta\dot{\alpha}(t)$$

$$= -k_1 x_1^2(t)$$

$$+ \Delta x_2(t)\left[u + x_1(t) - \frac{\partial x_{2,c}}{\partial x_1}\left(-k_1 x_1(t) + \Delta x_2(t)\right) + \varphi(x_1(t))\dot{\hat{\alpha}}(t)\right]$$

$$+ \Delta\alpha(t)\left[\underbrace{-x_1(t)\varphi(x_1(t)) + \frac{\partial x_{2,c}}{\partial x_1}\Delta x_2(t)\varphi(x_1(t)) + \gamma^{-1}\Delta\dot{\alpha}(t)}_{\text{want bracketted terms to vanish or be negative}}\right]$$

choose,

$$\begin{cases} u(t) = -x_1(t) - k_2\Delta x_2(t) + \dfrac{\partial x_{2,c}}{\partial x_1}\left(-k_1 x_1(t) + \Delta x_2(t)\right) + \varphi(x_1(t))\dot{\hat{\alpha}}(t) \\ \Delta\dot{\alpha}(t) = \dot{\hat{\alpha}}(t) = \gamma\left[x_1(t)\varphi(x_1(t)) - \dfrac{\partial x_{2,c}}{\partial x_1}\Delta x_2(t)\varphi(x_1(t))\right] \end{cases}$$

OR

$$\begin{cases} u(t) = -x_1(t) - k_2\Delta x_2(t) + \dfrac{\partial x_{2,c}}{\partial x_1}\left(x_2(t) + \hat{\alpha}(t)\varphi(x_1(t))\right) + \varphi(x_1(t))\dot{\hat{\alpha}}(t) \\ \Delta\dot{\alpha}(t) = \dot{\hat{\alpha}}(t) = \gamma\left[x_1(t)\varphi(x_1(t)) - \dfrac{\partial x_{2,c}}{\partial x_1}\Delta x_2(t)\varphi(x_1(t))\right] \end{cases}$$

• both version are equivalent. They just depend on how $\dot{x}_1(t)$ is expanded out to give $u(t)$.

$\Rightarrow$ with either selection

$$\dot{V} \leq -k_1 x_1^2(t) - k_2 \Delta x_2^2(t)$$

$\Rightarrow$ LaSalle - Yoshizawa

$$\left.\begin{array}{l} x_1(t) \to 0 \\ \Delta x_2(t) \to 0 \end{array}\right\} \text{ as } t \to \infty$$

$\Delta\alpha(t)$ bounded.

but note that the whole system is autonomous since there's no reference signal. That means we can use LaSalle's Theorem.

$\Rightarrow$

system stabilizes to the largest invariant set.

$\Rightarrow$

only option is the origin

$\Rightarrow$

$$x_1(t),\ \Delta x_2(t),\ \Delta\alpha(t) \to 0 \quad \text{as } t \to \infty$$

$\Rightarrow$ $x_{2,c}(t) \to 0$ as $t \to \infty$

$$x_1(t),\ x_2(t),\ \Delta\alpha(t) \to 0 \quad \text{as } t \to \infty$$

$\boxed{4}$ Complete control system

The complete adaptive backstepping state feedback system is

$$\begin{cases} u(t) = -x_1(t) - k_2\left[x_2(t) - k_1 x_1(t) - \hat{\alpha}(t)\varphi(x_1(t))\right] \\ \qquad - \left[k_1 + \hat{\alpha}(t)\,\partial\varphi(x_1(t))/\partial x_1\right]\left(x_2(t) + \hat{\alpha}\varphi(x_1(t))\right) - \varphi(x_1(t))\,\dot{\hat{\alpha}}(t) \\ \dot{\hat{\alpha}}(t) = \gamma\left[x_1(t)\varphi(x_1(t)) + \left(k_1 + \hat{\alpha}(t)\,\partial\varphi(x_1(t))/\partial x_1\right)\left(x_2(t) + \hat{\alpha}\varphi(x_1(t))\right)\varphi(x_1(t))\right] \end{cases}$$