## Adaptive Control w/ Input Constraints.

- Most practical systems have actuator limitations, arbitrary control authority is not possible. Can the system still achieve stabilization?

  If so, how? ... and can it be proven?

The typical setup goes as follows (back to scalar case):

$$\dot{x}(t) = a\,x(t) + b\,u(t)$$

input subject to constraints.

$$u(t) = u_{max}\ \mathrm{sat}\left(\frac{u_c(t)}{u_{max}}\right) = \begin{cases} u_c(t) & \text{if } |u_c(t)| \leq u_{max} \\ u_{max}\,\mathrm{sign}(u_c(t)) & \text{else} \end{cases}$$

actual command applied

commanded control input

saturation function: $\mathrm{sat}(\alpha) = \begin{cases} \alpha & |\alpha| \leq 1 \\ \mathrm{sign}(\alpha) & \text{else} \end{cases}$

$\Rightarrow$

PLANT: $x(t) = a\,x(t) + b u_c(t) + b\,\Delta u(t)$

$\Delta u(t) \equiv u(t) - u_c(t)$

control deficiency.

- To achieve proveable stabilization, we'll try to come up with an adaptive system that can compensate for the control deficiency.

The direct MRAC system uses:

CONTROLLER: $u_c(t) = k_x(t)\,x(t) + k_r(t)\,r(t)$

$\Rightarrow$

$$\dot{x}(t) = (a + bk_x(t))\,x(t) + bk_r(t)\,r(t) + b\Delta u(t)$$

this is new term in the system.
we cannot expect to track $x_m(t)$ if control to $x(t)$ can be bounded, but control to $x_m(t)$ need not be.

NOTE: Since the system cannot perform as desired, the demanded performance will have to be modified/adjusted.

$\Rightarrow$ modify model dynamics with an adaptive component so that it results in an achievable trajectory.

MODEL: $\dot{x}_m(t) = a_m x(t) + b_m[r(t) + k_u(t)\,\Delta u(t)]$ $\qquad a_m < 0$

this is an adaptive gain.
PROBLEM: now, not sure if model is a stable system when $\Delta u(t) \neq 0$.

additional parameter also adds one more matching condition,

$$b_m k_u^* = b$$

$\Rightarrow$

$$b k_r^* k_u^* = b$$

$\Rightarrow$

$$k_r^* k_u^* = 1$$

The model is trying to adapt to the control deficiency. If $b$ is precisely known, then $k_u^*$ is known & we can set $k_u(t) = k_u^* = b/b_m$, no adaptation necessary.

now that things are setup let's do the error & Lyapunov analysis.

Define error & error dynamics,

$$e(t) \equiv x(t) - x_m(t)$$

$$\dot{e}(t) = a_m e(t) + b\Delta k_x(t) x(t) + b\Delta k_r(t) r(t) - b_m \Delta k_u(t) \Delta u(t)$$

with
$$\Delta k_x(t) = k_x(t) - k_x^*$$
$$\Delta k_r(t) = k_r(t) - k_r^*$$
$$\Delta k_u(t) = k_u(t) - k_u^*$$

Use the candidate Lyapunov function

$$V(e(t), \Delta k_x(t), \Delta k_r(t), \Delta k_u(t))$$
$$= e^2(t) + |b|\left(\gamma_x^{-1} \Delta k_x^2(t) + \gamma_r^{-1} \Delta k_r^2(t)\right) + \gamma_u^{-1} \Delta k_u^2(t)$$

$\Rightarrow$

$$\dot{V} = 2e(t)\left[a_m e(t) + b\Delta k_x(t) x(t) + b\Delta k_r(t) r(t) - b_m \Delta k_u(t) \Delta u(t)\right]$$
$$+ 2\gamma_x^{-1} |b| \Delta k_x(t) \Delta\dot{k}_x(t) + 2\gamma_r^{-1} |b| \Delta k_r(t) \Delta\dot{k}_r(t) + 2\gamma_u^{-1} \Delta k_u(t) \Delta\dot{k}_u(t)$$

$\Rightarrow$

$$\dot{V} = -2|a_m| e^2(t) + 2|b| \Delta k_x(t)\left[x(t) e(t) \operatorname{sign}(b) + \gamma_x^{-1} \Delta\dot{k}_x(t)\right]$$
$$+ 2|b| \Delta k_r(t)\left[r(t) e(t) \operatorname{sign}(b) + \gamma_r^{-1} \Delta\dot{k}_r(t)\right]$$
$$+ 2\Delta k_u(t)\underbrace{\left[-\Delta u(t) e(t) b_m + \gamma_u^{-1} \Delta\dot{k}_u(t)\right]}_{\text{bracketted terms should vanish}}$$

$\Rightarrow$

$$\Delta\dot{k}_x(t) = \dot{k}_x(t) \equiv -\gamma_x\, x(t)\, e(t)\, \mathrm{sign}(b)$$

$$\Delta\dot{k}_r(t) = \dot{k}_r(t) \equiv -\gamma_r\, r(t)\, e(t)\, \mathrm{sign}(b)$$

ADAPTATION

$$\Delta\dot{k}_u(t) = \dot{k}_u(t) \equiv \gamma_u\, \Delta u(t)\, e(t)\, b_m$$

$\Rightarrow$

$$\dot{V} = -2|a_m|e^2(t) \leq 0$$ negative semi-definite

$\Rightarrow$

Lyapunov stable

$\Rightarrow$

$e(t)$, $\Delta k_x(t)$, $\Delta k_r(t)$, $\Delta k_u(t)$ all bounded

Although all above quantities are bounded, nothing can be concluded about $x(t)$ & $x_m(t)$ individually since $\Delta u(t)$ is "unknown." All we know is that $e(t) = x(t) - x_m(t)$ is bounded.

- How can we analyze the system to get stability?
  - We will have to constrain the system somehow since saturation prevents achievement of stability for all situations. So... for what situations can we reach some conclusion?

    ↳ a local analysis can lead to positive results.

    if we start close enough to stable point (origin), we may be able to show boundedness of either $x(t)$ or $x_m(t)$

    $\Rightarrow$ boundedness of all signals. Barbalat will finish it off!

Analysis of $x_m(t)$ made difficult through dependence on $x(t)$. On the other hand $x(t)$ does not depend on $x_m(t)$, so let's analyze $x(t)$ and hope that we can figure something out.

Choose the candidate Lyapunov function

$$W(x(t)) = \frac{1}{2} x^2(t)$$

if no saturation ever : $\dot{x}(t) = a\,x(t) + b\,u_c(t)$ & all is good.

if saturated needed : $\dot{x}(t) = a\,x(t) + b\,u_{max}\,\mathrm{sign}(u_c(t))$ & more analysis is needed.

$\Rightarrow$ saturation case

$$\dot{W} = a x^2(t) + b x(t)\, u_{max}\, \mathrm{sign}(u_c(t))$$

$$= a x^2(t) + u_{max}\, |b x(t)|\, \mathrm{sign}(u_c(t))\, \mathrm{sign}(b x(t))$$

At this point, there are two options to consider (as related to a)

I. system is stable naturally $(a < 0)$.

II. system is not stable.

Let's examine the two cases and see what can be concluded.

I. System naturally stable, $a < 0$.

in this case,

$$\dot{W} = -|a|\, x^2(t) + u_{max} |b x(t)| \, \text{sign}(u_c(t)) \, \text{sign}(b x(t))$$

$\Rightarrow$

$$\dot{W} \leq 0 \quad \text{if} \quad |x(t)| > \frac{u_{max} |b|}{|a|} = u_{max} \left| \frac{b}{a} \right|$$

$\Rightarrow$

since $W \geq 0$ on a compact domain & $\dot{W} \leq 0$ outside of it, we can conclude boundedness of $x(t)$.

$\Rightarrow$

$x_m(t)$ is bounded

$\Rightarrow$

Barbalat is used to get $e(t) \to 0$ as $t \to \infty$.

II. System is not stable

looking at $\dot{W}$, there are two subcases

i) $\text{sign}(u_c(t))\, \text{sign}(b x(t)) = -1$ ok

ii) $\text{sign}(u_c(t))\, \text{sign}(b x(t)) = 1$ not so ok.

In case (i), we get

$$\dot{W} = a x^2(t) - u_{max} |b x(t)|$$

$\Rightarrow$

$$\dot{W} \leq 0 \quad \text{if} \quad |x(t)| \leq u_{max} \left| \frac{b}{a} \right|$$

↑ opposite inequality as case I.

In case (ii), we are not so lucky. But, when does this occur?

$$\dot{W} = a x^2(t) + u_{max} |b x(t)| \geq 0$$

$\Rightarrow$ RECALL: $|u_c(t)| > u_{max}$ when in saturation mode

$$0 \leq \dot{W} \leq a x^2(t) + |u_c(t)|\,|b x(t)|$$

$\Leftrightarrow$ $u_c(t)$ & $bx(t)$ have the same sign.

$$0 \leq a x^2(t) + u_c(t)\, b\, x(t)$$

$\Leftrightarrow$

$$0 \leq a x^2(t) + [k_x(t) x(t) + k_r(t) r(t)]\, b x(t)$$

$\Leftrightarrow$

$$\leq [a + b k_x(t)]\, x^2(t) + b k_r(t) r(t) x(t)$$

$\Leftrightarrow$ by MATCHING ASSUMPTION $a = -|a_m| - b k_x^*$, plus add & subtract $k_r^*$

$$0 \leq [-|a_m| + b \Delta k_x(t)]\, x^2(t) + b\, [k_r^* + \Delta k_r(t)]\, r(t) x(t)$$

$\Rightarrow$

$$\leq [-|a_m| + |b| \overline{\Delta k_x}]\, x^2(t) + |b|\, [|k_r^*| + \overline{\Delta k_r}]\, r_{max} |x(t)|$$

max value, since bounded. $\overline{\Delta k_x} \equiv \max |\Delta k_x(t)|$

$r_{max} = \max |r(t)|$

$\Leftrightarrow$

$$|x(t)| \left[ (|a_m| - |b| \overline{\Delta k_x})\, |x(t)| - |b| (|k_r^*| + \overline{\Delta k_r})\, r_{max} \right] \leq 0$$

$\Leftrightarrow$

$$(|a_m| - |b| \overline{\Delta k_x}) \left[ |x(t)| - \frac{|k_r^*| + \overline{\Delta k_r}}{|a_m| - b \overline{\Delta k_x}} |b|\, r_{max} \right] \leq 0$$

Notice that the first inequality of this page implies satisfaction of the last inequality. It leads to a condition on $|x(t)|$ that must hold when $\dot{W} \geq 0$ under case (ii).

If we can determine the sign of $|a_m| - |b|\overline{\Delta k_x}$, then we can make the resulting condition on $|x(t)|$ more explicit.

So, how do we figure out if $|a_m| - |b|\overline{\Delta k_x} > 0$ ?

Let's just enforce it. Equivalently, enforce the inequality

$$\sqrt{V(0)} \leq \sqrt{\frac{|b|}{\gamma_x}} \; \frac{|a_m| - |k_r^*||a|\frac{r_{max}}{u_{max}}}{|b| + \alpha |a| \frac{r_{max}}{u_{max}}}, \qquad \text{for } \alpha > 1 \qquad (+)$$

$\Rightarrow$

$$\overline{\Delta k_x} \leq \frac{|a_m| - |k_r^*||a|\frac{r_{max}}{u_{max}}}{|b| + \alpha |a| \frac{r_{max}}{u_{max}}} \qquad (*)$$

$\Rightarrow$

$$|a_m| - |b|\overline{\Delta k_x} \geq |a|\frac{r_{max}}{u_{max}}\left(|k_r^*| + \alpha\overline{\Delta k_x}\right) \geq 0$$

Therefore, case (ii) implies that

$$|x| \leq \frac{|k_r^*| + \overline{\Delta k_r}}{|a_m| - b\overline{\Delta k_x}} |b| r_{max}$$

if $V(0)$ satisfies the above inequality, (+).

( if $V(0)$ satisfies inequality (+) and $|x(t)|$ does not satisfy the inequality, then case (ii) cannot hold.)

Manipulating (*), we get

$$\overline{\Delta k_x} \leq \frac{|a_m|\,u_{max} - |k_r^*|\,|a|\,r_{max}}{|b|\,u_{max} + \alpha\,|a|\,r_{max}}$$

$$\Leftrightarrow$$

$$\alpha\,|a|\,r_{max}\,\overline{\Delta k_x} + |b|\,u_{max}\,\overline{\Delta k_x} \leq |a_m|\,u_{max} - |k_r^*|\,|a|\,r_{max}$$

$$\Leftrightarrow$$

$$|a|\,r_{max}\left(\alpha\,\overline{\Delta k_x} + |k_r^*|\right) \leq \left(|a_m| - |b|\,\overline{\Delta k_x}\right) u_{max}$$

$$\Leftrightarrow$$

$$\frac{\alpha\,\overline{\Delta k_x} + |k_r^*|}{|a_m| - |b|\,\overline{\Delta k_x}}\, r_{max} \leq \frac{u_{max}}{|a|}$$

$\Rightarrow$ since $\alpha > 1$

$$\underbrace{\frac{\overline{\Delta k_x} + |k_r^*|}{|a_m| - |b|\,\overline{\Delta k_x}}\,|b|\,r_{max}} < \underbrace{u_{max}\left|\frac{b}{a}\right|}$$

recall that this was bound on $|x(t)|$ implied by being in case (ii) and inequality (†)

inequality bound on $|x(t)|$ if we are in case (i) and $\dot{W} \leq 0$.

All of this work leads to

$$\dot{W}(x(t)) < 0 \qquad \forall\, x(t) \in A \equiv \left\{ x : \frac{|k_r^*| + \overline{\Delta k_r}}{|a_m| - |b|\overline{\Delta k_x}} |b| r_{max} \leq |x| \leq u_{max} \left|\frac{b}{a}\right| \right\}$$

when $\Delta u(t) \neq 0$.

$\Rightarrow$

As long as the initial conditions start us inside of the annulus $A$, then the system state remains bounded. (Really inside $B(u_{max}|\frac{b}{a}|)$.)

$\Rightarrow$

found conditions on $x(t)$ to guarantee boundedness of $x(t)$.

$\Rightarrow$ since $e(t)$ bounded

$x_m(t)$ is bounded too!

$\Rightarrow$ Barbalat on $e(t)$, $\dot{e}(t)$

$e(t) \to 0$ as $t \to \infty$.

Theorem. Let $a, b, u_{max}$, and $r_{max}$ for the adaptive system be such that

$$\left|\frac{a_m}{b_m}\right| > \left|\frac{a}{b}\right| \frac{r_{max}}{u_{max}} .$$

how related.

first ensures positivity of term in second inequality.

If the system and the candidate Lyapunov function initial conditions satisfy

$$|x(0)| \le u_{max} \left|\frac{b}{a}\right|$$

and

$$\sqrt{V(0)} \le \sqrt{\frac{|b|}{\gamma_x}} \; \frac{|a_m| - |k_r^*|\,|a| \frac{r_{max}}{u_{max}}}{|b| + \alpha\, |a| \frac{r_{max}}{u_{max}}}$$

then the adaptive system with input constraints has a bounded solution for all $r(t)$ such that $r(t) \le r_{max}$. Furthermore, the error $e(t)$ goes to zero asymptotically.

Comments on all of this work:

1) if the system is naturally stable, then stability and error convergence is global.
2) unstable local.
3) neutrally stable global.

↳ for neutrally stable case $a = 0 \Rightarrow u_{max} \left|\frac{b}{a}\right| = \infty$

$\;\;\&\;\; \left|\frac{a}{b}\right| = 0$.