Nonlinear Stability

Patricio A. Vela

Georgia Institute of Technology

Dynamical Systems and Equilibria

Let's consider analyzing the following system:

$$\dot{x} = f(x,t), \quad x(0) = x_0, x \in \mathbb{R}^n, t \in \mathbb{R} \tag{1}$$

Note that we are divorcing ourself from control at the moment.

Definition 1 *A state x_0 is an equilibrium if (1), if once $x(t') = x_e$, then $x(t) = x_e$ for all $t \geq t'$ of the integral solution.*

Definition 2 *An equilibrium x_e is an isolated equilibrium if there exists an $r > 0$ such that the neighborhood $\mathcal{N}(x_e; r)$ has no other equilibria.*

Equivalent to the equilibrium condition is: $\dot{x}_e = f(x_e, t) = 0$ for all $t \geq t'$.

Stabillity

Definition 3 *The equilibrium state* x_e *is* <u>*stable*</u> *(in the sense of Lyapunov) if, for arbitrary* t_0 *and* $\epsilon > 0$*, there exists a* $\delta(\epsilon, t_0)$ *such that* $||x_0 - x_e|| < \delta$ *implies that* $||x(t; x_0, t_0) - x_e)|| < \epsilon$ *for all* $t \geq t_0$*.*

- If given an arbitrary neighborhood of the equilibrium, there exists a second neighborhood such that all trajectories starting the the second neighborhood also stay inside the given neighborhood forever.

Instability

Definition 4 *The equilibrium state x_e is unstable if it is not stable.*

- There exists a neighborhood of the the equilibrium such that, for any given neighborhood, there exists a trajectory starting in the given neighborhood that is gauranteed to leave the original neighborhood.

From here, things go nuts because there are many ways to be stable.

Uniform and Asymptotic Stability

Definition 5 *The equilibrium state x_e is uniformly stable (u.s., US) if, for arbitrary t_0 and $\epsilon > 0$, there exists a $\delta(\epsilon)$ such that $||x_0 - x_e|| < \delta$ implies that $||x(t; x_0, t_0) - x_e|| < \epsilon$ for all $t \geq t_0$.*

- Basically, the δ-neighborhood is independent of intial time.

Definition 6 *The equilibrium state is asymptotically stable (a.s., AS) if it is stable and there exists a $\delta(t_0)$ such that $||x_0 - x_e|| < \delta(t_0)$ implies that $\lim_{t\to\infty} ||x(t; x_0, t_0) - x_e|| = 0$.*

- Doesn't say what happens on the way, just that all trajectories starting out close enough to the equilibrium must ultimately approach the equilibrium.

Uniform Asymptotic Stability

Definition 7 *The equilibrium state is* *<u>uniformly asymptotically stable</u>* *(u.a.s., UAS) if it is uniformly stable and for all* $\delta > 0$ *and* $t_o \in \mathbb{R}^+$*, there is a* $\delta_0 > 0$ *and a* $T(\epsilon) > 0$ *such that* $||x_0 - x_e|| < \delta_0$ *implies that* $||x(t; x_0, t_0) - x_e|| < \epsilon$ *for all* $t \geq t_0 + T(\epsilon)$*.*

- Note that δ_0 is independent of ϵ and t_0, while T is independent of t_0.
- For any given neigborhood of the equilibrium, there exists a second neighborhood and a special time such that starting in the second neighborhood, I am gauranteed to enter the given neighborhood by the special time (and stay there).
- This really does implicitly lead to asymptotic stability.

Exponential Stability

Definition 8 *The equilibrium state x_e is <u>exponentially stable</u> (e.s.,E.S.) if there exists an $\alpha > 0$ such that for all $\epsilon > 0$, there is a $\delta(\epsilon) > 0$ such that $||x_0 - x_e|| < \delta(\epsilon)$ implies that $||x(t; x_0, t_0) - x_e|| \leq \epsilon e^{-\alpha(t-t_0)}$ for all $t \geq t_0$.*

- So, not only is the system asymptotically stable, but the rate at which is approaches the equilibrium is exponentially fast.
- Exponential stability can be tested in many way, one of which is to demonstrate linearity of the log-norm error versus time in a plot. (linearity of $\ln ||x(t; x_0, t_0) - x_e||$).

Global Stability and More ...

The adjective “global” can be attached to any of the above definitions if the δ-neighborhood can be arbitrarily large. Thus it is possible to obtain, globally stable, globally asymptotically stable, globally exponentiall stable, etc.

There are analagous definitions for instability, ...

and there are several other forms of stability (see textbooks on nonlinear dynamical systems theory).

Boundedness

Sometimes a system can be shown to be unstable about an equilibrium, yet it does not necessarily blow-up or demonstrate any similar form of horrendous behavior. In fact, its trajectories stay bounded for all time. In that case ...

Definition 9 *A solution to the system (1) is* <u>*bounded*</u> *if there exists a positive constant* c*, independent of* t_0*, such that* $||x(t; x_0, t_0)|| < c$ *for all* $t \geq t_0$*, where* c *may depend on the solution.*

This definition applies to a specific, chosen solution to the differential equation from (1). As with stability, we may define boundedness in terms of neighborhoods and not individual trajectories.

Boundedness of a System

Definition 10 *Solutions to the system (1) are*

- *uniformly bounded if there exists a positive constant c, independent of t_0, such that for all $a \in (0, c)$, there exists a $\beta(a) > 0$, independent of t_0, such that $||x(t_0)|| \leq a$ implies that $||x(t)|| < \beta(a)$ for all $t \geq t_0$.*
- *globally uniformly bounded if it is uniformly bounded for arbitrary c.*
- *uniformly ultimately bounded with ultimate bound b, if there exists positive constants b and c, independent of t_0, such that for all $a \in (0, c)$, there exists a $T = T(a, b)$, independent of t_0, such that $||x(t_0)|| \leq a$ implies that $||x(t)|| \leq b$ for all $t \geq t_0 + T$.*
- *globally uniformly ultimately bounded if it is uniformly ultimately bounded for arbitrary c.*

Note: boundedness and ultimate boundedness are equivalent.

Stability vs. Boundedness

- Stability is necessarily defined with respect to an equilibirum (or a limit set), whereas boundedness is not.

- Stability implies that I can stay arbitrarily close to an equilibrium point by starting even closer to it. This is too strong of a condition for systems experiencing unknown disturbances.

- For ultimate boundedness, the bound β (equivalent to the ϵ in stability definition) cannot be made arbitrarily small by starting closer to the equilibrium. In practical systems, the bound β depends on the disturbances and system uncertainties.