Invariant Set Theorems

Sometimes, we may find cases where Lyapunov's Direct Method is not satisfying. Perhaps we can show stability, but fail to show anything stronger and yet know that a stronger result should be possible.

As an example, let's consider the pendulum. Although local exp. stability can be shown using the linearization, only stability can be shown using Lyapunov's Direct Method.

pendulum:
$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -\frac{d}{mr^2} x_2 - \frac{g}{r}\sin(x_1)$$

candidate Lyapunov function is energy : $V(x) = \frac{1}{2}x_2^2 + \frac{g}{r}(1-\cos(x_1))$

$\Rightarrow$

$$\frac{d}{dt}V(x) = x_2\dot{x}_2 + \frac{g}{r}\sin(x_1)\dot{x}_1$$
$$= -\frac{d}{mr^2}x_2^2 - \cancel{\frac{g}{r}\sin(x_1)\,x_2} + \cancel{\frac{g}{r}\sin(x_1)\,x_2}$$
$$= -\frac{d}{mr^2}x_2^2$$

$\Rightarrow$

$\dot{V} \leq 0$ it is $\underbrace{\text{negative semi-definite}}$

semi-definite because $\dot{V}=0$ on the x_1-axis (anytime $x_2=0$) and not uniquely at the origin.

$\Rightarrow$

can only conclude local stability.
surely there's a way to conclude asymptotic stability!

Well, we can ① try harder to find a better V.

- this could be a mission impossible.

② see what alternative methods exist.

Let's check out the second option.

We are considering the following autonomous system,

$$\dot{x} = f(x) \qquad x(0) = x_0 \tag{$*$}$$

where $f: \mathbb{R}^n \to \mathbb{R}$ is Lipschitz cts on $\mathbb{R}^n$.

Definition. Given a solution $x(t)$ to the IVP $(*)$, a point $q \in \mathbb{R}^n$ is called a <u>positive limit point</u> (or <u>accumulation point</u>) of $x(t)$ if there exists a sequences of times $\{t_n\}$ with $t_n \to \infty$ as $n \to \infty$, such that $x(t_n) \to q$ as $n \to \infty$. The set of all positive limit points is called the <u>positive limit set</u>.

Definition. A set Ω is called an <u>invariant set</u> with respect to $(*)$ if

$$x(0) \in \Omega \;\Rightarrow\; x(t) \in \Omega \qquad \forall\, t \in \mathbb{R}.$$

A set Ω is called <u>positively invariant</u> if

$$x(0) \in \Omega \;\Rightarrow\; x(t) \in \Omega \qquad \forall\, t \geq 0.$$

Definition. A closed invariant set $\Omega \subset \mathbb{R}^n$ is called an <u>attracting set</u> if there exists a neighborhood $N(\Omega)$ such that

$$\forall\, x_0 \in N(\Omega),\ t \geq 0, \quad x(t; x_0) \in N(\Omega) \text{ and } \lim_{t \to \infty} x(t; x_0) \to \Omega$$

Definition. A closed, connected set M is a <u>trapping region</u> if it is positively invariant or, equivalently, the dynamics (*) on ∂M point inwards.

∂M = boundary of M.

- Defining and showing the existence of a Lyapunov function for an equilibrium is equivalent to finding a trapping region for the equilibrium. The following theorems loosen the conditions on the trapping region.

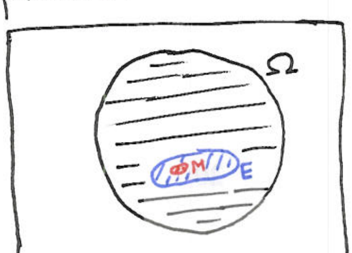
Theorem [LaSalle's Invariance Principle]

Let $\Omega \subset D \subset \mathbb{R}^n$ be a compact set that is positively invariant with respect with respect to (*). Let $V: \mathbb{R}^n \to \mathbb{R}$ be such that $V|_D \in C^1(D; \mathbb{R})$ and $\dot{V}|_\Omega \leq 0$. Let E be the set of all points in Ω where $\dot{V} = 0$ and let M be the largest invariant set contained in E. Then every solution starting in Ω converges to M as $t \to \infty$.

- If $D = \mathbb{R}^n$ and V is radially unbounded with $\dot{V} \leq 0 \quad \forall x(t) \in \mathbb{R}^n$, then every solution starting in $\mathbb{R}^n$ converges to M. (global version)

- It is essential that (*) be autonomous.

Visualization of spaces as sets:



$M \subset E \subset \Omega \subset D \subset \mathbb{R}^n$

- If the goal is to show that $x(t) \to 0$ as $t \to \infty$, then must show that $M = \{0\}$, e.g., the largest invariant set contains the origin.

Theorem [Barbashin - Krasovskii].

Let $x=0$ be an equilibrium point for (*) and $V: \mathbb{R}^n \to \mathbb{R}$ be $V|_D \in C^1(D; \mathbb{R})$ with $\dot{V}(x(t)) \leq 0 \;\forall\, x(t) \in D$, where $D \subset \mathbb{R}^n$ contains the origin. Let $S = \{ x \in D \mid \dot{V}(x(t)) = 0 \}$ and suppose that no other solution can stay identically in S other than the trivial solution $x(t) \equiv 0$. Then the origin is asymptotically stable.

- If $D = \mathbb{R}^n$ is possible and V is radially unbounded, then the origin is globally asymptotically stable.

* these are not the only Invariant Set theorems, but they do give a good flavor as to what's going on.

Examples.

1 Pendulum: If we continue the analysis, we get that

$$\dot{V} \leq 0 \quad \text{on the set } E = \{ (x_1, x_2) \subset \Omega \mid x_2 = 0 \}$$

where Ω is a set local to the origin.

Looking at the definition of V, $V = \frac{1}{2} x_2^2 + \frac{g}{r}(1-\cos(x_1))$

we see that V pos. def. about the origin holds for $-\pi < x_1 < \pi$,

and let's restrict x_2 to be not too large.

$\Rightarrow$

$$\Omega = \{ (x_1, x_2) \mid -\pi < x_1 < \pi \text{ and } -\alpha < x_2 < \alpha \}$$

for some α.

On the set E, we have that

$$\dot{x} = f(x) = \begin{Bmatrix} 0 \\ -\frac{g}{r}\sin(x_1) \end{Bmatrix} \Rightarrow \dot{x}_2 \neq 0 \text{ if } x_1 = 0$$

$\Rightarrow$

only the origin stays in E, of all points in E.

$\Rightarrow$

$$M = \{0\}$$

$\Rightarrow$ LaSalle / Barbashin-Krasovskii

The origin is asymptotically stable.

2] This example is like the pendulum.

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -x_1 + (x_1^2 - 1)\, x_2$$

Even though linearization can show stability of the origin, the goal of giving this example is to demonstrate how the Invariant Set Theorems fill in a gap that Lyapunov's direct method cannot handle.

Given $V = \frac{1}{2}(x_1^2 + x_2^2)$

$\Rightarrow$

$$\dot{V} = x_1 \dot{x}_1 + x_2 \dot{x}_2$$
$$\dot{V} = x_1 x_2 - x_1 x_2 + (x_1^2 - 1)\, x_2^2 \leq 0 \quad \text{inside the unit circle.}$$

$\Rightarrow$

Lyapunov's direct method ensures stability and nothing else.

Asymptotic stability cannot be concluded because $\dot{V}$ is negative semi-definite.

(as opposed to negative definite)

Lets try to understand where we have $\dot{V} = 0$.

$$E = \{(x_1, x_2) \mid \dot{V} = 0\}$$
$$= \{(x_1, x_2) \mid x_1 = \pm 1 \text{ OR } x_2 = 0\}$$

$\Rightarrow$ restrict to ~~[illegible]~~ inside the unit circle

$$E = \{(x_1, x_2) \in \mathcal{N}(1) \mid x_2 = 0\}$$

2] This example is like the pendulum.

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -x_1 + (x_1^2 - 1)x_2$$

for $V(x) = \frac{1}{2}(x_1^2 + x_2^2)$

$$\dot{V} = x_1\dot{x}_1 + x_2\dot{x}_2$$
$$= x_1x_2 - x_1x_2 + (x_1^2-1)x_2^2 = (x_1^2-1)x_2^2 \leq 0 \quad \text{for} \quad x_1^2 + x_2^2 < 1$$

can only conclude stability of origin inside the unit ball around the origin, $\Omega = \{(x_1, x_2) \mid x_1^2 + x_2^2 < 1\}$.

Trajectories are readily seen to be stable, so ...

$$E = \{(x_1, x_2) \in \Omega \mid x_2 = 0\}$$

examine $f|_E$

$$f|_E = \begin{Bmatrix} 0 \\ -x_1 \end{Bmatrix} \Rightarrow x_2 \neq 0 \text{ if } x_1 \neq 0.$$

$$M = \{0\}$$

$\Rightarrow$ Barbashin-Krasovskii

The origin is (locally) asymptotically stable.

] Recall the adaptive controller introduced in the first week

$$\dot{x} = ax + u \qquad x(0) = x_0$$
$$u = -\hat{k}x$$
$$\dot{\hat{k}} = \gamma x^2 \qquad \hat{k}(0) = 0, \ \gamma > 0$$

$\Rightarrow$

$$\dot{x} = (a - \hat{k})x \qquad x(0) = x_0$$
$$\dot{\hat{k}} = \gamma x^2 \qquad k(0) = 0 \ , \ \gamma > 0.$$

choose $V = \frac{1}{2}x^2 + \frac{1}{2\gamma}(\hat{k} - b)^2$ for $b > a$

$\Rightarrow$

$$\begin{aligned} \dot{V} &= x\dot{x} + \frac{1}{\gamma}(\hat{k} - b)\dot{\hat{k}} \\ &= (a - \hat{k})x^2 + (\hat{k} - b)x^2 \\ &= (a - b)x^2 \end{aligned}$$
$$\dot{V} = -(b - a)x^2 \leq 0$$

$\dot{V}$ is negative semi-definite.

We have $E = \{ (x, \hat{k}) \in \Omega_\beta \mid x = 0 \}$ as our invariant set

where $\Omega_\beta = \{ (x, \hat{k}) \in \mathbb{R}^n \mid V(x, \hat{k}) < \beta \}$

these sets are compact and positively invariant by construction.
(for any $\beta < \infty$ since V is radially unbounded)

The largest invariant subset of E under the system dynamics is E itself

$\Rightarrow$

$M = E$

$\Rightarrow$ LASALLE

All trajectories in $\Omega_\beta \to E$ as $t \to \infty$

$\Rightarrow$ hold for any β

$$x(t) \to 0 \quad \text{as} \quad t \to \infty \qquad \forall\, x_0 \in \mathbb{R}^n$$

$\Rightarrow$

x dynamics are globally asymptotically stable.

* Cannot conclude stability for $\hat{k}$, but since Ω_β is compact and invariant, we can conclude boundedness of $\hat{k}$.

] Now, we'll consider a less trivial example.

$$\dot{x} = y + x[1-x^2-y^2]$$
$$\dot{y} = -x + y[1-x^2-y^2]$$

linearization about the origin gives instability, but the following analysis shows some interesting behavior.

define $V = \frac{1}{2}(x^2+y^2)$

$\Rightarrow$

$$\dot{V} = x\dot{x} + y\dot{y}$$
$$= xy + x^2[1-x^2-y^2] - xy + y^2[1-x^2-y^2]$$
$$= (x^2+y^2)[1-x^2-y^2]$$

$$\dot{V} > 0 \quad \text{if} \quad x^2+y^2 < 1$$
$$\dot{V} = 0 \quad \text{if} \quad x^2+y^2 = 1$$
$$\dot{V} < 0 \quad \text{if} \quad x^2+y^2 > 1$$

Note also that we can write

$$\dot{V} = \nabla V \cdot f = (x,y)\cdot(\dot{x},\dot{y}) = 0 \quad \text{on} \quad x^2+y^2=1$$

$\Rightarrow$

velocity vector $\perp$ state vector on unit circle

$\Rightarrow$

looks like unit circle is an invariant set
(possibly even stable if we look at when $\dot{V}<0$ and $\dot{V}>0$)

But, how do we show this since V is not a valid Lyapunov function in the sense of having a negative (semi) definite time derivative.

Use a different V. Try the following

$$V = \frac{1}{4}(1-x^2-y^2)^2$$

$\Rightarrow$

$$\begin{aligned}\dot{V} &= -(1-x^2-y^2)(x\dot{x}+y\dot{y}) \\ &= -(1-x^2-y^2)(x^2+y^2)(1-x^2-y^2) \\ &= -(x^2+y^2)(1-x^2-y^2)^2 \\ &= -(x^2+y^2)V\end{aligned}$$

note that the invariant set $E = \{(x,y) \mid x^2+y^2 = 1\}$ defines when $\dot{V}$ vanishes. Since it is invariant under f (the dynamics), we have $M = E$.

$\Rightarrow$ LASALLE & radial unboundedness

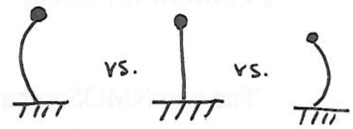
The unit circle is an asymptotically stable invariant set.

Otherwise put, the unit circle is an attracting set.

] Buckling Column

$\dot{x}_1 = x_2$

$\dot{x}_2 = -a x_2 + b x_1 - c x_1^3$



$\Rightarrow$

fixed points are $(0,0)$, $(\sqrt{\frac{b}{c}}, 0)$, $(-\sqrt{\frac{b}{c}}, 0)$

will use Lyapunov & Chaeter Theorems to show stability/instability.

a] the origin.

if we set $x_2 = 0$, then $\dot{x}_1 = 0$

$\dot{x}_2 = b x_1 - c x_1^3 > 0$ if $0 < x_1 < \sqrt{\frac{b}{c}}$ flow is up

< 0 if $-\sqrt{\frac{b}{c}} < x_1 < 0$ flow is down

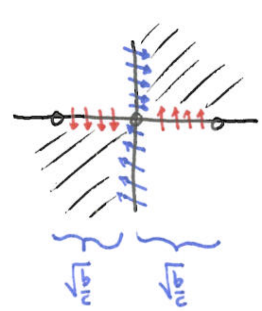
if we set $x_1 = 0$, then $\dot{x}_1 = x_2$

$\dot{x}_2 = -a x_2$

if $x_2 > 0$ flow is right & down

$x_2 < 0$ flow is left & up

$\Rightarrow$



$\rightarrow$ want to examine behavior of flow in vicinity of origin in quadrants I and III.

if $V(x_1, x_2) = x_1 \cdot x_2$, then it is positive in quadrants I & III and zero on the axes.

if we check $\dot{V}$, we get

$$\dot{V} = \dot{x}_1 x_2 + x_1 \dot{x}_2 = x_2^2 + x_1(-ax_2 + bx_1 - cx_1^3)$$

$$= x_2^2 - ax_1x_2 + bx_1^2 - cx_1^4$$

$$= x_2^2 + cx_1^2\left(\tfrac{b}{c} - x_1^2\right) - ax_1x_2$$

if $a=0$, then we'd have $\dot{V} \geq 0$ pos. definite inside the circle $N(\sqrt{\frac{b}{c}})$ centered at the origin. Excluding the origin, $\dot{V} > 0$.

Taking $D = N(\sqrt{\frac{b}{c}})$ and $\Omega = \{ (x_1, x_2) \in N(\sqrt{\frac{b}{c}}) \mid x_1 \cdot x_2 > 0 \}$

Chaeter's Thm shows instability.

if $a > 0$, then we need, $V = x_1x_2 + \frac{1}{2}ax_1^2$

and it's possible to show that $\dot{V} = x_2^2 + cx_1^2\left(\frac{b}{c} - x_1^2\right) \geq 0$

inside $N(\sqrt{\frac{b}{c}})$. Excluding the origin $\dot{V} > 0$.

$\Rightarrow$

the function $V = x_1x_2 + \frac{1}{2}ax_1^2$ can be used to show instability of the origin.

b] the other points.

for starters choose $x_e = \left(\sqrt{\frac{b}{c}}, 0\right)$

translate it to the origin

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -a x_2 + b x_1 - c x_1^3$$

$$\dot{y}_1 = y_2$$
$$\dot{y}_2 = -a y_2 + b\left(y_1 + \sqrt{\tfrac{b}{c}}\right) - c\left(y_1 + \sqrt{\tfrac{b}{c}}\right)^3$$

choose $V = \frac{1}{2} y_2^2 - \frac{1}{2} b\left(y_1 + \sqrt{\frac{b}{c}}\right)^2 + \frac{1}{4} c\left(y_1 + \sqrt{\frac{b}{c}}\right)^4 + \frac{1}{4}\frac{b^2}{c}$

$\Rightarrow$

$$\dot{V} = y_2 \dot{y}_2 - b\left(y_1 + \sqrt{\tfrac{b}{c}}\right)\dot{y}_1 + c\left(y_1 + \sqrt{\tfrac{b}{c}}\right)^3 \dot{y}_1$$

$$= -a y_2^2 + \cancel{b\left(y_1 + \sqrt{\tfrac{b}{c}}\right)^1 y_2} - \cancel{c\left(y_1 + \sqrt{\tfrac{b}{c}}\right)^3 y_2} - \cancel{b\left(y_1 + \sqrt{\tfrac{b}{c}}\right)^1 y_2} + \cancel{c\left(y_1 + \sqrt{\tfrac{b}{c}}\right)^3 y_2}$$

$$\dot{V} = -a y_2^2 \leq 0$$

and

$$E = \left\{ (y_1, y_2) \in \mathcal{N}\left(\sqrt{\tfrac{b}{c}}\right) \middle| \dot{V} = 0 \right\} = \left\{ (y_1, y_2) \in \mathcal{N}\left(\sqrt{\tfrac{b}{c}}\right) \middle| y_2 = 0 \right\}$$

but, the only invariant point in E is the origin.

$\Rightarrow$

$$M = \{0\}$$

$\Rightarrow$ Barbashin - Kravoskii

The origin is asymptotically stable.

- the same type of analysis holds for $x_e = \left(-\sqrt{\frac{b}{c}}, 0\right)$.