DYNAMICAL SYSTEMS THEORY

Dynamical systems analyzed as systems of differential equations:

$$\dot{x}(t) = f(x(t), t), \qquad x(t_0) = x_0, \tag{1}$$

where

$x \in \mathbb{R}^n$ is the <u>state vector</u>,

$t \in [t_0, \infty)$ is <u>time</u>,

$x_0 \in \mathbb{R}^n$ is the <u>initial condition</u>,

and

$f: \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ are the <u>state dynamics</u> or <u>system dynamics</u>.

- models the underlying physics of the problem at hand.

Since dynamical systems theory relies on some ideas from Analysis, we'll go over some preliminaries...

Definition (Continuity). A function $f: \mathbb{R} \to \mathbb{R}$ is continuous at a point $x_0 \in \mathbb{R}$, if for any $\epsilon > 0$, there exists a $\delta(\epsilon; x_0)$ such that if $|x - x_0| \le \delta(\epsilon; x_0)$, then $|f(x) - f(x_0)| \le \epsilon$.

Definition (Uniform Continuity). A function $f: \mathbb{R} \to \mathbb{R}$ is uniformly continuous in the domain $D \subset \mathbb{R}$ if for any $\epsilon > 0$, there exists a $\delta(\epsilon)$ such that for any $x, y \in D$ if $|x - y| \le \delta(\epsilon)$ then $|f(x) - f(y)| \le \epsilon$.

Abbreviations: cts - continuous.

unif. - uniformly.

Definition (Differentiable). A function $f:\mathbb{R}\to\mathbb{R}$ is differentiable at $x_0\in\mathbb{R}$ if the limit

$$f'(x_0) = \lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h}$$

exists.

Definition (Continuously differentiable). A function $f:\mathbb{R}\to\mathbb{R}$ is continuously differentiable at $x_0\in\mathbb{R}$ if it is continuous at x_0, as is its derivative at x_0.

Notation: $C^0(\mathbb{R};\mathbb{R})$ - space of continuous functions from $\mathbb{R}$ to $\mathbb{R}$.

$C^1(\mathbb{R};\mathbb{R})$ - space of continuously differentiable functions from $\mathbb{R}$ to $\mathbb{R}$.

⋮

$C^k(\mathbb{R};\mathbb{R}) \leftarrow$ continuously differentiable to k^{th} order

⋮

$C^\infty(\mathbb{R};\mathbb{R}) \leftarrow$ called space of smooth functions from $\mathbb{R}$ to $\mathbb{R}$.

Definition. (locally Lipschitz continuous). A function $f:\mathbb{R}\to\mathbb{R}$ is locally Lipschitz continuous at $x_0\in\mathbb{R}$ if for all $x,y\in N(x_0)$, $\exists$ an $L>0$ such that

$$\underbrace{|f(x)-f(y)| \le \overset{\text{Lipschitz constant}}{L}\,|x-y|}_{\text{Lipschitz condition}}$$

for some neighborhood $N(x_0)$.

Definition (Lipschitz continuous). A function $f: \mathbb{R} \to \mathbb{R}$ is Lipschitz continuous in the domain $D \subset \mathbb{R}$ if [redacted]

it is locally Lipschitz continuous for all $x_0 \in D$.

Definition (uniformly Lipschitz continuous). A function $f: \mathbb{R} \to \mathbb{R}$ is uniformly Lipschitz continuous in $D \subset \mathbb{R}$ if $\exists L > 0 : \forall x, y \in D$, $|f(x) - f(y)| \leq L|x - y|$.

- here there is a universal constant L such that the Lipschitz condition is gauranteed for all pairs of points in D.

- if the domain can be chosen so that $D = \mathbb{R}$, then the function is globally Lipschitz continuous.

Examples.

1] domain matters. take tangent function

tan $\quad D = [-\pi/2, \pi/2]$ only locally Lipschitz at points (all except for $\pm\pi/2$)

$D = (-\pi/2, \pi/2)$ Lipschitz cts in D

$D = [-\pi/2 + \epsilon, \pi/2 - \epsilon]$ uniformly Lipschitz in D.

$\frac{\pi}{2} > \epsilon > 0$

2] to see that space of Lipschitz cts $\supset$ space of cts. diff.

$|x|$ is Lipschitz cts on $\mathbb{R}$, but not cts. diff.

why? not differentiable at $x = 0$.

$\boxed{3}$ to see that space of unif. cts $\supset$ space of Lipschitz cts

$x^{1/3}$ is uniformly continuous on any compact subset $D \subset \mathbb{R}$ containing the origin. It is not Lipschitz cts on D because the derivative blows up,

$$\frac{d}{dx} x^{1/3} = \frac{1}{3} x^{-2/3},$$

at the origin. More loosely, it is cts on $\mathbb{R}$ but not Lipschitz cts on $\mathbb{R}$.

These examples are leading to the following observations

- the Lipschitz condition can be rewritten

$$\frac{|f(x)-f(y)|}{|x-y|} \leq L$$

leading to the fact: if a function has a bounded derivative at x, then it is locally Lipschitz cts at x.

- As far as the relative sizes of these spaces is concerned:

$$\text{space of cts functions} \supset \text{space of Lipschitz cts functions} \supset \text{space of cts. diff. functions}$$

Abbreviations & Notations:

diff. - differentiable

loc. - local | locally

$\exists$ - there exists

$\forall$ - for all

- piecewise: when the property holds everywhere except for a finite number of locations
- globally: when the property holds over the entire domain of definition (typically $\mathbb{R}$).

These concepts can be extended to $\mathbb{R}^n$ and $\mathbb{R}^m$, $f: \mathbb{R}^n \to \mathbb{R}^m$, so long as norms are defined on both spaces $\mathbb{R}^n$ and $\mathbb{R}^m$, $n, m > 0$.

Definition. A norm on $\mathbb{R}^n$ is a function $\|\cdot\| : \mathbb{R}^n \to \mathbb{R}$ such that

1) $\|x\| \geq 0 \quad \forall\, x \in \mathbb{R}^n$.

2) $\|x\| = 0$ iff $x = 0$

3) $\|\lambda x\| = |\lambda|\, \|x\| \quad \forall\, \lambda \in \mathbb{R},\ x \in \mathbb{R}^n$

4) $\|x+y\| \leq \|x\| + \|y\| \quad \forall\, x, y \in \mathbb{R}^n$

Norms can be induced. Let's see how this works for the space of linear maps between $\mathbb{R}^n$ and $\mathbb{R}^m$, $L(\mathbb{R}^n; \mathbb{R}^m)$.

An element $f \in L(\mathbb{R}^n; \mathbb{R}^m)$ is defined by $f(x) \equiv Ax$ for some A.

$\Rightarrow$

if ~~a~~ norms are defined on $\mathbb{R}^n$ & $\mathbb{R}^m$, then let $y = f(x) = Ax$.

The norm of f is

$$\|f\| = \|A\| = \max_{\underbrace{\|x\| = 1}_{\text{norm on } \mathbb{R}^n}} \underbrace{\|Ax\|}_{\text{norm on } \mathbb{R}^m}$$

- like finding the direction of maximal amplification under A, then using amplification factor as the value of the norm.

normally, a function $f: \mathbb{R}^n \to \mathbb{R}^m$ is bounded if $\exists c>0 : \|f(x)\| < c \quad \forall x \in \mathbb{R}^n$.

What happens when we have $f: \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^m$, a function of two variables?

- our notion of boundedness must account for that

- a function $f(x,t)$ is <u>bounded at t_0</u> if $\exists c>0 : \|f(x,t_0)\| < c \quad \forall x \in \mathbb{R}^n$
- a function $f(x,t)$ is <u>uniformly bounded</u> if $\exists c>0 : \forall t \in [t_0,t_1],\ \|f(x,t)\| \le c$.

The next two Lemmas relate Lipschitz continuity with differentiability under specific conditions.

Lemma. Let $f(x,t) : D \times [t_0,t_1] \to \mathbb{R}^n$ be cts on $D \subset \mathbb{R}^n$. Suppose that $D_1 f(x,t)$ exists and is continuous on some compact subset $W \subset D$ and that the Jacobian is uniformly bounded, e.g., $\exists L>0$:

$$\|D_1 f(x,t)\| \le L \qquad \forall (x,t) \in W \times [t_0,t_1].$$

Then,

$$\|f(x,t) - f(y,t)\| \le L\|x-y\| \quad \forall x,y \in W \text{ and } t \in [t_0,t_1].$$

$\to$ f is uniformly Lipschitz on $W \times [t_0,t_1]$ (in x).

Lemma. If $f(x,t)$ and $D_1 f(x,t)$ are cts on $\mathbb{R}^n \times [t_0,t_1]$, then f is globally Lipschitz in x iff $D_1 f$ is uniformly bounded on $\mathbb{R}^n \times [t_0,t_1]$.

Notation: iff - if and only if

$D_1 f(x,t) = \frac{\partial f}{\partial x}(x,t)$ and $D_2 f(x,t) = \frac{\partial f}{\partial t}(x,t)$

So, how does this relate to (1)?

- Problem defined by (1) is an Initial Value Problem (IVP).

  Goal is to find solution to (1). We want/need for it to be unique.

Definition. A continuous function $x: [t_0, t_1] \to \mathbb{R}^n$, satisfying $x(t_0) = x_0$, is called a solution of (1) over $t \in [t_0, t_1]$ if $\dot{x}(t)$ is defined $\forall\, t \in [t_0, t_1]$ and $\dot{x}(t) = f(x(t), t)$ $\forall\, t \in [t_0, t_1]$.

- a solution called also be called a <u>state trajectory</u> or a <u>system trajectory</u>.

- NOTICE: <u>a</u> solution and not <u>the</u> solution.

Example. $\dot{x}(t) = \sqrt{x(t)}$, $x(0) = 0$, $x \in \mathbb{R}^+$, $t \geq 0$.

$\Rightarrow$ two solutions: 1) $x(t) \equiv 0$

2) $x(t) = \frac{1}{4} t^2$

(there are really an ∞ of solutions)

Theorem (Cauchy/Peano Existence Theorem). If $f(x,t)$ is cts in a closed neighborhood of x_0, $\overline{N}(x_0, t_0; R, T)$, then there exists a $\delta < T$ such that the IVP has at least one cts solution $x(t)$ for $t_0 < t < t_0 + \delta$.

Notation: $\overline{N}(x_0, t_0; R, T) \equiv \{ x, t \mid |x - x_0| \leq R \text{ and } |t - t_0| \leq T \}$

- the above example is cts as needed by the Existence Theorem.

Theorem (Local Existence and Uniqueness). If $f(x,t)$ is piecewise cts in t and locally Lipschitz cts at x_0, then $\exists\, \delta > 0$: the dynamical system in (1) has a unique solution for $t \in [t_0, t_0+\delta]$.

Example. $\dot{x}(t) = \sqrt{x(t)}$, $x(0) = x_0 \neq 0$, $x \in \mathbb{R}^+$, $t \geq 0$

Then, $x(t) = \frac{1}{4}(t + 2\sqrt{x_0})^2$ on $t \in [0, \delta]$ for some $\delta > 0$.

Example. $\dot{x}(t) = x^2(t)$, $x(0) = 1$, $t \geq 0$.

Solution is $x(t) = -\dfrac{1}{t-1}$. It exists only for finite time.

The maximal possible δ is 1, at which point there is blow-up.

really only defined on $[0, \delta)$ for $0 < \delta \leq 1$.

(equivalently $[0, \delta]$ for $0 < \delta < 1$)

- If a solution exists for all time, then the solution is called complete.

Theorem (Global Existence and Uniqueness). If $f(x,t)$ is piecewise cts in t and globally Lipschitz in $\mathbb{R}^n$ for x, then the IVP (1) is complete; a unique solution exists for $[t_0, \infty)$.

- this theorem is a bit conservative.

Example. As an example of the conservativeness of the prior Theorem, consider,

$$\dot{x}(t) = -x^3(t), \qquad x(0) = x_0,$$

which has the unique solution

$$x(t) = \frac{x_0}{\sqrt{2x_0^2 t + 1}}.$$

It is complete for any initial condition $x_0 \in \mathbb{R}^n$, but it is not globally Lipschitz.

Theorem (Global Existence and Uniqueness on a Compact Domain).
Let $f(x,t)$ be piecewise continuous in t, ~~locally~~ uniformly Lipschitz in $D \subset \mathbb{R}^n$ for all $t \geq 0$ and let $W \subset D$ compact such that $x_0 \in W$. Suppose that every solution to the IVP lies entirely in W, then there is a complete, unique solution evolving on $[0,\infty)$.