

sources of optical flow (relevant):

- relative motion between object & viewer
 ↳ allows to infer spatial arrangements
- discontinuities in flow field can be used for segmentation
- motion recovery (objects & or self)
- shape

violates: uniform sphere w/shading

- ① rotate sphere
- ② move light source.

Horn & Schunck version:

$$\nabla I \cdot X = -I_t$$

Component / Magnitude of movement in the direction of the brightness gradient equals

$$-\frac{I_t}{\|\nabla I\|_2}$$

cannot determine component in direction of iso-brightness contours.

add smoothness constraint $\min \lambda (\|\nabla u\|^2 + \|\nabla v\|^2)$

$$\begin{aligned} &\|\nabla u\|^2 \\ &u_x^2 + u_y^2 \\ &\|\nabla(u + \delta u)\|^2 \\ &\Rightarrow 2 \nabla u \cdot \nabla \delta u \end{aligned}$$

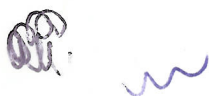
⇒

$$\mathcal{L}(X) = \iint \left[(\nabla I \cdot X + I_t)^2 + \alpha^2 (\|\nabla u\|^2 + \|\nabla v\|^2) \right] dx dy$$

⇒ compute 1st variation:

$$\frac{\delta \mathcal{L}}{\delta X} \cdot \delta X = \iint \left[2(\nabla I \cdot X + I_t) \nabla I \cdot \delta X + \alpha^2 (2 \nabla u \cdot \nabla \delta u + 2 \nabla v \cdot \nabla \delta v) \right] dx dy$$

$$= \iint \left[2(\nabla I \cdot X + I_t) \nabla I \cdot \delta X + 2\alpha^2 (\nabla u \cdot \nabla \delta u + \nabla v \cdot \nabla \delta v) \right] dx dy$$



In Matlab,

\bar{u}, \bar{v} are obtained by convolution.

$(\alpha^2 + I_x + I_y)$ is a matrix
 I_x, I_y, I_t are matrices } constant once
 images are given

~~I_x, I_y, I_t~~

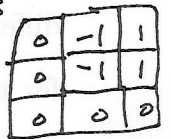
$I_x \bar{u}, I_y \bar{v}$ Computed matrices.

quiver \leftarrow tell them how it works!

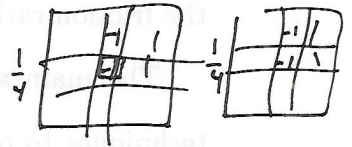
need two images, α^2 parameter, and # iterations.



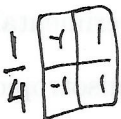
only one for loop in MATLAB
 optical flow function.



can use basic I_x, I_y, I_t
 as can get fancier



$H \frac{1}{4} S$:
$$I_x = \frac{1}{4} \left(\underbrace{I(i, j+1, k)} - \underbrace{I(i, j, k)} + \underbrace{I(i+1, j+1, k)} - \underbrace{I(i+1, j, k)} \right. \\ \left. + \underbrace{I(i, j+1, k+1)} - \underbrace{I(i, j, k+1)} + \underbrace{I(i+1, j+1, k+1)} - \underbrace{I(i+1, j, k+1)} \right)$$



I_y is similar

$$I_t = \frac{1}{4} \left(\underbrace{I(i, j, k+1)} - \underbrace{I(i, j, k)} + \underbrace{I(i+1, j, k+1)} - \underbrace{I(i+1, j, k)} \right. \\ \left. + \underbrace{I(i, j+1, k+1)} - \underbrace{I(i, j+1, k)} + \underbrace{I(i+1, j+1, k+1)} - \underbrace{I(i+1, j+1, k)} \right)$$



average of forward Euler in space and time.

by parts

$$\iint \nabla u \cdot \nabla \delta u \, dx \, dy$$

$$\left. \nabla u \cdot \nabla \delta u \right|_{\partial \Omega} - \iint \Delta u \cdot \delta u$$

$$\begin{aligned} \frac{\delta L}{\delta X} \cdot \delta X &= \iint \left[2(\nabla I \cdot X + I_t) \nabla I \cdot \delta X - 2\alpha^2 (\Delta u, \Delta v) \cdot \delta X \right] dx \, dy \\ &= \iint \left[2(\nabla I \cdot X + I_t) \nabla I - 2\alpha^2 (\Delta u, \Delta v) \right] \cdot \delta X \, dx \, dy \end{aligned}$$

two components in integrand must vanish:

$$2(\nabla I \cdot X + I_t) I_x - 2\alpha^2 \Delta u = 0$$

$$2(\nabla I \cdot X + I_t) I_y - 2\alpha^2 \Delta v = 0$$

⇒

$$I_x^2 u + I_y^2 v = \alpha^2 \Delta u - I_x I_t$$

$$I_x I_y u + I_y^2 v = \alpha^2 \Delta v - I_y I_t$$

⇒ def. of Laplacian in paper

$$(\alpha^2 + I_x^2) u + I_x I_y v = \alpha^2 \bar{u} - I_x I_t$$

$$I_x I_y u + (\alpha^2 + I_y^2) v = \alpha^2 \bar{v} - I_y I_t$$

⇒

$$\begin{bmatrix} \alpha^2 + I_x^2 & I_x I_y \\ I_x I_y & \alpha^2 + I_y^2 \end{bmatrix} \begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{Bmatrix} \alpha^2 \bar{u} - I_x I_t \\ \alpha^2 \bar{v} - I_y I_t \end{Bmatrix}$$

$$\alpha^2 (\alpha^2 + I_x^2 + I_y^2) \begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{bmatrix} \alpha^2 + I_y^2 & -I_x I_y \\ -I_x I_y & \alpha^2 + I_x^2 \end{bmatrix} \begin{Bmatrix} \alpha^2 \bar{u} - I_x I_t \\ \alpha^2 \bar{v} - I_y I_t \end{Bmatrix}$$

⇒

$$\begin{aligned} u &= (\alpha^2 + I_x^2 + I_y^2)^{-1} \left[(\alpha^2 + I_y^2) \bar{u} - I_x I_y \bar{v} - I_x I_t \right] \\ v &= (\alpha^2 + I_x^2 + I_y^2)^{-1} \left[(\alpha^2 + I_x^2) \bar{v} - I_x I_y \bar{u} - I_y I_t \right] \end{aligned}$$

$$\alpha^4 + \alpha^2 I_x^2 - \alpha^2 I_y^2$$

$$\alpha^2 (\alpha^2 + I_x^2 + I_y^2)$$

define $\Delta u, \Delta v$

⇒ ALTERNATIVELY

~~$$(\alpha^2 + I_x^2 + I_y^2)(u - \bar{u}) = -$$~~

$$u = \bar{u} - (\alpha^2 + I_x^2 + I_y^2)^{-1} [I_x (I_x \bar{u} + I_y \bar{v} + I_t)]$$

$$v = \bar{v} - (\alpha^2 + I_x^2 + I_y^2)^{-1} [I_y (I_x \bar{u} + I_y \bar{v} + I_t)]$$

this is the solution.

but since we need to find these u , we use gradient descent.

called Gauss-Seidel & is used for linear systems.

note that solution to u is linear in \bar{u} and \bar{v}

⇒

if

$$x = Ay$$

solution given by

x

$$r_k = (x_k - Ay_k)$$

↑ this is defect.

$$Ax = 0$$

$$Ax_k = r_k$$

$$x_{k+1} = x_k + r_k = x_k - (x_k - Ay_k) = Ay_k$$

but then y changes implicitly

$$\Rightarrow r_{k+1} = x_{k+1} - Ay_{k+1}$$

⇒

$$x_{k+1} = Ay_k$$

$$u^{n+1} = \bar{u}^n - I_x (\alpha^2 + I_x^2 + I_y^2)^{-1} (I_x \bar{u}^n + I_y \bar{v}^n + I_t)$$

$$v^{n+1} = \bar{v}^n - I_y (\alpha^2 + I_x^2 + I_y^2)^{-1} (I_x \bar{u}^n + I_y \bar{v}^n + I_t)$$

in

In optical flow case, we have

$$Ax = b$$

→

$$(I + N)x = b$$

$$N = I - A$$

→

$$Ix_{n+1} = b + Nx_n$$

↑ image only terms
 ↑ OFC + smoothness

But, we can do this differently.

$$\begin{aligned} u^{n+1} &= u^n - \frac{I_x \bar{u}^n + I_y \bar{v}^n + I_t}{1 + \alpha^2 (I_x^2 + I_y^2)} I_x \\ v^{n+1} &= v^n - \frac{I_x \bar{u}^n + I_y \bar{v}^n + I_t}{1 + \alpha^2 (I_x^2 + I_y^2)} I_y \end{aligned} \quad \left. \begin{array}{l} I_x \\ I_y \end{array} \right\} \text{in class}$$

↖ in this other case, we have

$$E = \alpha^2 E_{\text{OFC}} + E_s$$

* discuss boundary conditions!

$$(u_x, u_y)^T \cdot \hat{n} = 0 \quad (v_x, v_y)^T \cdot \hat{n} = 0$$

• the two algorithms are equivalent, but can differ.

↖ need to verify this with my code.
 ‡ did I program both?

optical flow field:

velocity field ~~of an~~ defined on image domain that transforms one image to another

→ not uniquely determined (projection equations one problem)

motion field:

projection onto the image of three-dimensional vectors

other problem sources of error:

specular effects

shadows

insufficient texturing

occlusion

define a vector field?

$$X: \mathbb{R}^2 \rightarrow \mathbb{R}^2$$

↑

at every point
in image

↑

defines a vector
describing particle
motion

if we integrate the vector field

$$\dot{x} = X(x), \quad x(0) = x_0$$

then we get particle trajectories for various x_0 . Call them $\Phi_{0,t}^X(x_0)$

but, at an edge where there is more image variation, the situation is less ambiguous.

furthermore, all particles at an edge move with ~~the~~ locally consistent velocities.

what's one way to impose local consistency?
 e.g. restrict local variation

isotropic smoothing e.g. reg smoothness

because the net flow effect on the director is re-normalized by backflow. Figure 2(a) shows the in-phase component of the director $(A \cdot E_x)$ as a function of the dimensionless radial distance (r) for dimensionless frequencies (ω) : 0.1, 1, 10, 100, 1000, and 10 000. $T = 24^\circ C$, and the reactive parameter: 0.2325, in the nonaligning regime. Figure 2(b) shows the corresponding out-of-phase component of the orientation $(A \cdot E_y)$ as a function of the dimensionless radial distance (r) . The in-phase component decreases monotonically with frequency, while the out-of-phase component exhibits resonance behavior that slightly exceeds maximum elastic storage. The behavior in the aligning regime is obtained by reversing the sign of the amplitude.

B. Velocity field

Since the director field n is coupled to the velocity field v , imposing an oscillatory pressure drop to the NLC will produce a velocity field with in-phase and out-of-phase components. Thus the total dimensionless velocity field (v_x, v_y) is given by the sum of the following in-phase and out-of-phase components:

$$v_x(r, \omega) = v_x^i(r, \omega) \sin(\omega t) + v_x^o(r, \omega) \cos(\omega t) \quad (22)$$

Using eq (22) and separation of variables, the in-phase (v_x^i, v_y^i) and out-of-phase (v_x^o, v_y^o) director components are found to be

where $h_1(r)$ and $h_2(r)$ are the Kelvin functions of order ν .

$$h_1(r) = \sum_{k=0}^{\infty} \frac{\cos\left(\frac{3\pi}{4}\nu + \frac{\pi}{2}k\right)}{k! \Gamma(k+1) + \nu} \left(\frac{r}{2}\right)^{2k+\nu} \quad (18)$$

$$h_2(r) = \sum_{k=0}^{\infty} \frac{\sin\left(\frac{3\pi}{4}\nu + \frac{\pi}{2}k\right)}{k! \Gamma(k+1) + \nu} \left(\frac{r}{2}\right)^{2k+\nu} \quad (19)$$

The amplitude of the director field δ is a function of ω . The symmetry and scaling of the amplitudes are

$$\delta_x^i(\omega, \delta_1^i) = -\delta_x^o(\omega, -\delta_1^o), \quad \delta_y^i(\omega, \delta_1^i) = -\delta_y^o(\omega, -\delta_1^o) \quad (20)$$

$$\delta_x^i(\omega, \delta_1^i) = \delta_x^o(\omega, \delta_1^o), \quad \delta_y^i(\omega, \delta_1^i) = \delta_y^o(\omega, \delta_1^o) \quad (21)$$

Vanishing amplitudes are a signature of the alignment-orientation transition. The amplitude sign reversal indicates characteristic rheological responses. The only viscous case in the problem is the Maxwell velocity δ associated with the axially oriented director field [i.e. $\nu = (0, 0, 1)^T$] and the transient shear velocity δ associated with out-of-phase director around the axial ν axis. The frequency dependence of δ is weighted by the shear viscosity η .

Suppose an imaged particle is moving, then it satisfies the following equation

$$I(\Phi_{0,t}^X(x), t) = I(x, 0)$$

assuming nothing else funny is going on

called "Brightness Constancy Assumption"

⇒

time derivative vanishes

$$\frac{d}{dt} I(\Phi_{0,t}^X(x), t) = \frac{\partial}{\partial t} I(x, 0)$$

⇒

$$\nabla I \cdot \frac{\partial \Phi_{0,t}^X}{\partial t} + \frac{\partial I}{\partial t} = 0$$

⇒

$$\nabla I \cdot X + \frac{\partial I}{\partial t} = 0$$

⇒

$$\frac{\partial I}{\partial x} u + \frac{\partial I}{\partial y} v + \frac{\partial I}{\partial t} = 0$$

optical flow constraint equation (OFC)

Problems one constraint (OFC) versus two unknowns (u, v).

⇒

multiple solutions possible.

↳ equiconours



if intensity is uniform in a region, then vector field is a-biguous but should also be "uniform" in some sense.

Iterative Methods.

$$Ax = b$$



$$(S+N)x = b$$

invertible.

⇒

$$Sx = b + Nx$$

⇒

$$x = S^{-1}(b + Nx)$$

$$x_{n+1} = S^{-1}(b + Nx_n)$$

$$= S^{-1}b + S^{-1}Nx_n$$

in our case, structure of optical flow lets us solve this pixel by pixel
since S is the diagonal part of A and N is the off-diagonal part.

$$x_{k+1} = b + (S+N)x_k?$$

Richardson iteration

$$S = I, \quad N = (I - A)$$

OFC + L_2 smoothness → iterative method.

$$Ax^* = b$$

⇒

$$Sx^* = b + Nx^*$$

$$x^* = S^{-1}b + S^{-1}Nx^*$$

$$x^* = x - e$$

$$r = Ax - b$$

$$r = Ax - Ax^* = Ae$$

↑ true solution

⇒

$$x = A^{-1}r = x^*$$

don't have

$$r_k = Ax_k - Ax^*$$

$$r_{k+1} = Ax_{k+1} - Ax^*$$

$$= AS^{-1}(b + Nx_k) - Ax^*$$

$$= AS^{-1}b + AS^{-1}Nx_k - Ax^*$$

=

$$e_{k+1} = x_{k+1} - x^* = S^{-1}b + S^{-1}Nx_k - x^*$$

$$= S^{-1}Nx_k - S^{-1}Nx^*$$

$$= S^{-1}N e_k$$

all good, as long as $\|S^{-1}N\| < 1$

eig. of $S^{-1}N$ are less than unity

⇒

error will reduce w/ each iteration.

GRADIENT DESCENT OPTICAL FLOW:

instead of solving for the minimal solution, one can use gradient descent to arrive at it,

$$\frac{\delta u}{\delta t} = - \left[(\nabla I \cdot X + I_t) I_x - \alpha^2 \Delta u \right]$$

$$\frac{\delta v}{\delta t} = - \left[(\nabla I \cdot X + I_t) I_y - \alpha^2 \Delta v \right]$$

⇒

$$\frac{u^{n+1} - u^n}{\delta t} = (\nabla I \cdot X + I_t) I_x - \alpha^2 \Delta u$$

⇒

$$u^{n+1} = u^n - \delta t \left[(\nabla I \cdot X + I_t) I_x - \alpha^2 \Delta u \right]$$

$$v^{n+1} = v^n - \delta t \left[(\nabla I \cdot X + I_t) I_y - \alpha^2 \Delta v \right]$$

$$\Delta u = u^{++} + u^{+-} + u^{-+} + u^{--} - 4u$$

$$5u - u^{++}$$