

Curves and Curve Evolution:

→ a curve you can smooth using some optimization strategy but most state-of-the-art methods tend to just move the vertices to make it smoother.

Observation: the boundary of an object forms a closed curve.



part of Mumford-Shah energy functional seeks to minimize the length of the boundary

⇒ minimize length of closed curve.

* this is a problem in curve evolution theory.
to do this will require some preliminaries.

closed curve ad closed loop. Length is defined as the sum of lengths of edges.

ad length of edge ℓ is defined as the distance between its endpoints.

* we have a closed curve, defined by $C: \mathbb{S}^1 \rightarrow \mathbb{R}^2$

(takes the circle and maps it to curve in \mathbb{R}^2)

alternatively

$$C: [0, 1] \rightarrow \mathbb{R}^2$$

$$C: [0, L] \rightarrow \mathbb{R}^2$$

length of curve

some parametrization - arc-length parametrization.

uses Euclidean 2-norm, $s = \int_0^P \langle C_p, C_p \rangle^{1/2} dp$

which length of curve is $L = \int_0^1 \langle e_p, e_p \rangle^{1/2} dp = \int_0^1 \|e_p\| dp$

$$\text{or } L = \int_0^L ds$$

why since humans think that means you integrate it to get the curve's length? a gradient function gives you information about the local behavior of the function at a point. \Rightarrow the gradient function tells us that the gradient has been decreasing from left to right. so what's the relationship between the gradient and the derivative? $ds = \|e_p\| dp$ \Rightarrow the gradient function is the derivative of the function. "but why isn't it just the derivative?" because the derivative is a local measure of how much the function changes at a point. "but what's the derivative of the derivative?" \Rightarrow the second derivative is a local measure of how much the function changes at a point. Not that it's always true, but it's a good starting point.

note also that $\frac{ds}{dp} = \|e_p\|^{1/2}$
 \Rightarrow if we want to calculate the derivative of the function, we need to take the derivative of the function with respect to the parameter p . $\frac{1}{\|e_p\|^{1/2}} \frac{d}{dp} = \frac{d}{ds}$ \Rightarrow this is a local measure of how much the function changes at a point. \Rightarrow the second derivative is a local measure of how much the function changes at a point. \Rightarrow the second derivative is a local measure of how much the function changes at a point.

$$\frac{2}{dp} f(s) = f'(s) \frac{ds}{dp} = \|e_p\|^{1/2} f'(s) \Rightarrow \text{and } \frac{1}{\|e_p\|^{1/2}} \frac{2}{dp} f(s) = \frac{2}{ds} f(s)$$

$\frac{2}{ds} f(s) = f''(s)$ \Rightarrow we can relate their differentials to area functions.

Other properties, definitions

$$T = \frac{de}{ds} = \frac{1}{\|e_p\|} \frac{de}{dp} e_p, \frac{1}{\|e_p\|} e_p$$

$$N = \frac{dT}{ds} = \frac{\frac{d}{ds} \left(\frac{de}{dp} \right)}{\left\| \frac{de}{dp} \right\|} e_s, \quad \left\| \frac{dT}{ds} \right\| \equiv K$$

called curvature

$$\frac{dT}{ds} = K N$$

$$\frac{dN}{ds} = ?? \star T + ? N$$

$$\begin{aligned} & \langle T, T \rangle = 1 \\ \Rightarrow & \langle T, \frac{dT}{ds} \rangle = 0 \\ \Rightarrow & T \perp \frac{dT}{ds} \end{aligned}$$

$$L(C + \epsilon V) = \int_0^1 \|C_p + \epsilon V_p\| dp$$

\Rightarrow

$$\frac{\partial L}{\partial \epsilon} \Big|_{\epsilon=0} = \int_0^1 \frac{1}{2} \cdot 2 \cdot \frac{\langle C_p + \epsilon V_p, V_p \rangle}{\langle C_p + \epsilon V_p, C_p + \epsilon V_p \rangle} \Big|_{\epsilon=0} dp$$

$$\|C_p\| dp = ds$$

$$\frac{ds}{ds} = \frac{1}{\|C_p\|} \frac{ds}{dp}$$

$$= \int_0^1 \frac{\langle C_p, V_p \rangle}{\langle C_p, C_p \rangle} dp$$

$$\frac{d}{ds} \tau = KN$$

$$= \int_0^1 \frac{1}{\|C_p\|} \langle C_p, V_p \rangle dp$$

$$= \frac{\langle C_p, V_p \rangle}{\|C_p\|} \Big|_0^1 - \int_0^1 \frac{d}{dp} \frac{C_p}{\|C_p\|}, V \rangle dp$$

$$= - \int_0^L \frac{d}{dp} \frac{C_p}{\|C_p\|} \langle \frac{1}{\|C_p\|} \frac{1}{dp} \frac{C_p}{\|C_p\|}, V \rangle dp$$

$$= - \int_0^L \langle \frac{1}{ds} C_s, V \rangle ds \frac{1}{\|C_p\|} dp$$

$$C_s = \frac{1}{ds} \frac{1}{ds} C_p$$

$$= \frac{1}{\|C_p\|} \frac{1}{dp} \frac{1}{\|C_p\|} C_p$$

$$\frac{\partial L}{\partial \epsilon} \Big|_{\epsilon=0} = - \int_0^L \langle KN, V \rangle ds$$

$$\Rightarrow V = KN$$

$$= \frac{1}{\|C_p\|} \frac{1}{dp} \frac{1}{\|C_p\|} C_p$$

$$= \frac{1}{\|C_p\|^2} C_{pp} - \frac{1}{\|C_p\|} \frac{C_p C_{pp}}{\|C_p\|^3}$$

$$= \frac{C_{pp}}{\|C_p\|^2} - \frac{C_p C_{pp}}{\|C_p\|^3}$$

$$\Rightarrow \text{to shrink curve, } \frac{dC}{ds} = KN$$

$$C_r = KN$$

$$L = \int_0^1 \|c_p\| dp = \int_0^L \text{arc-length} ds \quad ds = \frac{1}{\|c_p\|} dp$$

$$L = \int_0^1 \langle c_p, c_p \rangle^{\frac{1}{2}} dp \quad \leftarrow \text{Euclidean 2-norm}$$

$$\Rightarrow L(C_p + \delta C) = \int_0^1 \langle C_p + \delta C_p, C_p + \delta C_p \rangle^{\frac{1}{2}} dp$$

$$\frac{\delta L}{\delta \sigma} \Big|_{\sigma=0} = \int_0^1 \frac{1}{2} \cdot 2 \cdot \frac{\langle \delta C_p, C_p + \delta C_p \rangle}{\langle C_p + \sigma \delta C_p, C_p + \sigma \delta C_p \rangle} \Big|_{\sigma=0} dp$$

$$= \int_0^1 \frac{\langle C_p, \delta C_p \rangle}{\langle C_p, C_p \rangle} dp$$

$$= \int_0^1 \frac{\langle C_p \rangle}{\|C_p\|} \frac{\langle C_p \rangle}{\|C_p\|} \delta C_p dp$$

$$= \frac{C_p}{\|C_p\|} \cdot \delta C \Big|_0^1 - \int_0^1 \left\langle \frac{d}{dp} \frac{C_p}{\|C_p\|}, \delta C \right\rangle dp$$

$$= - \int_0^1 \left\langle \frac{C_{pp}}{\|C_p\|} - \frac{\langle C_p, C_{pp} \rangle C_p}{\|C_p\|^3}, \delta C \right\rangle dp$$

$$\begin{aligned} \frac{d}{dp} \langle C_p, C_p \rangle^{\frac{1}{2}} \\ = -\frac{1}{2} \langle C_p, C_p \rangle^{-\frac{1}{2}} \cdot 2 \langle C_p, C_{pp} \rangle \end{aligned}$$

Suppose curve is a circle:

bottom right part, owing to which both curves are continuous along the curve, so that the curve is smooth.

It is the case that the radius vector is constant, so that the curve is a circle.

\Rightarrow and right now instead of this we can take the curve to be a circle with center at the origin and radius r .

$T = \frac{dC}{ds} = \left(-\sin\left(\frac{s}{r}\right), \cos\left(\frac{s}{r}\right) \right)$

$= \left(-\sin\left(\frac{s}{r}\right), \cos\left(\frac{s}{r}\right) \right)$

 **AS A VECTOR!**

$$KN = \frac{dT}{ds} = \left(-\frac{1}{r} \cos\left(\frac{s}{r}\right), -\frac{1}{r} \sin\left(\frac{s}{r}\right) \right) = \frac{1}{r} (-\cos\left(\frac{s}{r}\right), \sin\left(\frac{s}{r}\right))$$

$$= \frac{1}{r} K$$

\Rightarrow the curvature is constant.

$K = \frac{1}{r}$ (Curvature is constant).

descris inversely proportional to radius of circle

bottom right part, owing to which both curves are continuous along the curve, so that the curve is smooth.

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$= \left(-\sin\left(\frac{s}{r}\right), \cos\left(\frac{s}{r}\right) \right)$

curvature describes the "amount of bending" of curve at a point. it has local dependence on the curve (through derivatives)

What about minimizing area?

$$L = \int_R dA$$

from div. theorem, $\int_{\partial R} F \cdot N ds = \int_R \nabla \cdot F dA = \int_R f dA.$

so, define

$$\mathcal{E}(C) = \int_C F \cdot N ds$$

but for planar curves, $N = JT$ where $J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

$\Rightarrow \mathcal{E}(C) = \int_C F \cdot JT \bar{s} ds = \int_0^1 F \cdot JT C_p \frac{\|C_p\|}{\|C_p\|} d\mu = \int_0^1 F \cdot JT C_p d\mu$

$$\begin{aligned} \mathcal{E}(C + \epsilon V) &= \int_0^1 F \cdot JT (C_p + \epsilon V_p) d\mu \\ &= \int_0^1 [(DF \cdot V) \cdot JT (C_p + \epsilon V_p) + F \cdot JT V_p] d\mu \\ &= \int_0^1 [(DF \cdot V) \cdot JT C_p + F \cdot JT V_p] d\mu \\ &= \cancel{\int_0^1 F \cdot JT V_p} + \int_0^1 [(DF \cdot V) \cdot JT C_p - \frac{\partial}{\partial p}(F) \cdot JT V_p] d\mu \end{aligned}$$

$$= \int_0^1 [(DF \cdot V) \cdot JT C_p - (DF \cdot C_p) \cdot JT V_p] d\mu$$

$$= \int_0^1 [(JT C_p)^T DF \cdot V - (DF \cdot C_p) \cdot JT V] d\mu$$

$$= \int_0^l [(\mathbf{J} \mathbf{C}_p)^T \mathbf{D}\mathbf{F} - (\mathbf{D}\mathbf{F} \cdot \mathbf{C}_p)^T \mathbf{J}] \mathbf{V} \, d\mathbf{p}$$

$$= \int_0^l [\mathbf{C}_p^T (\mathbf{J} \mathbf{D}\mathbf{F} - \mathbf{D}\mathbf{F} \mathbf{J})] \mathbf{V} \, d\mathbf{p}$$

define $\phi \mathbf{J} = \mathbf{J}^T \mathbf{D}\mathbf{F} - \mathbf{D}\mathbf{F}^T \mathbf{J}$
 $\Rightarrow \phi = -\mathbf{J}^T \mathbf{D}\mathbf{F} \mathbf{J} \neq \mathbf{D}\mathbf{F}$

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial F^1}{\partial x^1} & \frac{\partial F^1}{\partial x^2} \\ \frac{\partial F^2}{\partial x^1} & \frac{\partial F^2}{\partial x^2} \end{bmatrix}$$

$$\begin{bmatrix} \frac{\partial F^2}{\partial x^1} & \frac{\partial F^2}{\partial x^2} \\ \frac{\partial F^1}{\partial x^1} & -\frac{\partial F^1}{\partial x^2} \end{bmatrix} = \begin{bmatrix} \frac{\partial F^2}{\partial x^2} & \frac{\partial F^1}{\partial x^1} \\ \frac{\partial F^2}{\partial x^1} & \frac{\partial F^1}{\partial x^2} \end{bmatrix}$$

$$= \int_0^l \mathbf{C}_p^T \phi \mathbf{J} \mathbf{V} \, d\mathbf{p}$$

$$= \int_0^l \phi \mathbf{J}^T \mathbf{C}_p \mathbf{V} \, d\mathbf{p}$$

$$= \int_0^l \phi \mathbf{J}^T \frac{\mathbf{C}_p}{\|\mathbf{C}_p\|} \mathbf{V} \, d\mathbf{p}$$

$$= \int_0^l \phi \mathbf{J}^T \mathbf{C}_s \mathbf{V} \, ds$$

$$= \int_0^L -\phi \mathbf{N} \mathbf{V} \, ds$$

$$= \int_0^L -\phi \langle \mathbf{N}, \mathbf{V} \rangle \, ds$$

$$= \int_0^L \langle -\phi \mathbf{N}, \mathbf{V} \rangle \, ds$$

\Rightarrow

$$\int_0^L \langle (\nabla \cdot \mathbf{F}) \mathbf{N}, \mathbf{V} \rangle \, ds$$

$$= \int_0^L \langle f \mathbf{N}, \mathbf{V} \rangle \, ds$$

$$= \int_C \langle f \mathbf{N}, \mathbf{V} \rangle \, ds$$

$$\phi = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

$$\phi = \begin{bmatrix} \frac{\partial F^1}{\partial x^1} & \frac{\partial F^2}{\partial x^1} \\ \frac{\partial F^2}{\partial x^1} & \frac{\partial F^2}{\partial x^2} \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial F^1}{\partial x^1} & \frac{\partial F^1}{\partial x^2} \\ \frac{\partial F^2}{\partial x^1} & \frac{\partial F^2}{\partial x^2} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} \frac{\partial F^1}{\partial x^2} & \frac{\partial F^1}{\partial x^1} \\ \frac{\partial F^2}{\partial x^2} & \frac{\partial F^2}{\partial x^1} \end{bmatrix}$$

$$+ \begin{bmatrix} -\frac{\partial F^2}{\partial x^2} & \frac{\partial F^2}{\partial x^1} \\ \frac{\partial F^1}{\partial x^2} & -\frac{\partial F^1}{\partial x^1} \end{bmatrix}$$

\Rightarrow

$$\phi = \begin{bmatrix} -\frac{\partial F^1}{\partial x^1} - \frac{\partial F^2}{\partial x^2} & 0 \\ 0 & -\frac{\partial F^1}{\partial x^1} - \frac{\partial F^2}{\partial x^2} \end{bmatrix}$$

\Rightarrow

$$\phi = -\nabla \cdot \mathbf{F}$$

but, what does this flow have anything to do with the Energy functional?

well, $\iint_D (I-J)^2 dA$ is one of the terms in the functional.

how do we deal w/ it? \rightarrow start w/ assumption of one region

\Rightarrow

$$\iint_R (I-J_{in})^2 dA + \iint_{D \setminus R} (I-J_{out})^2 dA$$

$\begin{cases} J_{in} & x \in R \\ J_{out} & x \notin R \end{cases}$

some boundaries, but different inward pointing normals.

\Rightarrow

$$\frac{\partial C}{\partial t} = [(I-J_{in})^2 - (I-J_{out})^2] N$$

? inward pointing normal?

so, we can handle this if J is fixed. remember when discussing the variation part in previous parts, we said that if J is constant, then the flow will move along the boundary. so we can handle this by fixing J . now, what about the case where J is not constant?

what about $\iint \phi(\|\nabla J\|) dA$? same deal

$$\frac{\partial C}{\partial t} = [\phi(\|\nabla J_{in}\|) - \phi(\|\nabla J_{out}\|)] N$$

so, we know how to deal w/ the Γ part.

now, what about the J part?

need to look at variations w/r respect to J .

$$\mathcal{E}(J, \Gamma) = \iint_D (I - J)^2 dA + \iint_{D \setminus \Gamma} \phi(\|\nabla J\|) dA + \text{~} \underbrace{\sim \int_{\Gamma} ds}_{|\Gamma|}$$

\downarrow
does not depend on J
 \Rightarrow ignore for now.

consider image part only, with Γ fixed.

$$\bar{\mathcal{E}}(J, \Gamma) = \iint_D (I - J)^2 dA + \iint_{D \setminus \Gamma} \phi(\|\nabla J\|) dA$$

$\underbrace{\quad \quad \quad}_{\text{do this first}}$

suppose we have a linear function

$$f(I, \epsilon) \quad \del{f(I + \epsilon f(I))} = f(I) + f(\epsilon I)$$

$$f(I) = \text{constant}$$

$$\bar{\mathcal{E}}_1(J + \epsilon \delta J, \Gamma) = \iint_D f^2(I - J - \epsilon \delta J) dA$$

$$\frac{\delta \bar{\mathcal{E}}_1}{\delta \epsilon} \Big|_{\epsilon=0} = - \iint_D 2 f(I - J - \epsilon \delta J) \cdot f'(I - J - \epsilon \delta J) \cdot \delta J dA$$

$$= - \iint_D (2 f(I - J) \cdot f'(I - J)) \cdot \delta J dA$$

$\underbrace{\quad \quad \quad}_{\text{this is minimizing solution}}$

$$2 f(I - J) \cdot f'(I - J) = 0$$

for our case

$$2(I - J) = 0$$

\Rightarrow

$$J = I$$

what's the deal here? Nothing, This part is correct,
but we forgot about the second term

$$\phi(x^2) = 2x$$

\Rightarrow

$$\frac{\delta \tilde{E}_2}{\delta \epsilon} \Big|_{\epsilon=0} = - \int_A \nabla \left[\frac{\phi'(\|\nabla J\|)}{\|\nabla J\|} \nabla J \right] \delta u \, dA$$

MISSING STEPS HERE!!!

\Rightarrow

$$\frac{\delta \tilde{E}}{\delta \epsilon} \Big|_{\epsilon=0} = \iint_R \left[-2(I-J_{in}) - 2\nabla \left[\frac{\phi'(\|\nabla J_{in}\|)}{\|\nabla J_{in}\|} \nabla J_{in} \right] \right] \delta u \, dA$$

$$+ \iint_D \left[-2(I-J_{out}) - 2\nabla \left[\frac{\phi'(\|\nabla J_{out}\|)}{\|\nabla J_{out}\|} \nabla J_{out} \right] \right] \delta u \, dA$$

\Rightarrow get critical point:

$$-(I - J_{in}) - \nabla \left[\frac{\phi'(\|\nabla J_{in}\|)}{\|\nabla J_{in}\|} \nabla J_{in} \right] = 0$$

$$-(I - J_{out}) - \nabla \left[\frac{\phi'(\|\nabla J_{out}\|)}{\|\nabla J_{out}\|} \nabla J_{out} \right] = 0$$

classical definition of ϕ is $\phi(x) = x^2 \Rightarrow \phi' = 2x$

\Rightarrow

$$-I + J_{in} - \nabla [\nabla J_{in}] = 0 \quad w/ \langle \nabla J_{in}, N \rangle = 0$$

\Rightarrow

$$-I + J_{in} - \Delta J_{in} = 0 \quad w/ \langle \nabla J_{in}, N \rangle = 0$$

let's check out the second term. (we may need to specify further!)

$$\iint_{\partial\Gamma} \phi(\|\nabla J\|) dA = \iint_R \phi(\|\nabla J_{in}\|) dA + \iint_{\partial R} \phi(\|\nabla J_{out}\|) dA$$

OK, so important thing is to get $\iint_A \phi(\|\nabla J\|) dA$

$$\frac{\delta \bar{E}_2}{\delta \epsilon} \Big|_{\epsilon=0} = \bar{E}_2(J + \epsilon \nabla \delta J, \Gamma) = \iint_A \phi(\|\nabla J + \epsilon \nabla \delta J\|) dA$$

$$\frac{\delta \bar{E}_2}{\delta \epsilon} \Big|_{\epsilon=0} = \iint_A \phi'(\|\nabla J + \epsilon \nabla \delta J\|) \cdot \left. \frac{\langle \nabla \delta J, \nabla J + \epsilon \nabla \delta J \rangle}{\|\nabla J + \epsilon \nabla \delta J\|} \right|_{\epsilon=0} dA$$

\Rightarrow

$$\frac{\delta \bar{E}_2}{\delta \epsilon} \Big|_{\epsilon=0} = \iint_A \phi'(\|\nabla J + \cancel{\epsilon \nabla \delta J}\|) \cdot \frac{\langle \nabla \delta J, \nabla J \rangle}{\|\nabla J + \cancel{\epsilon \nabla \delta J}\|} dA$$

\Rightarrow

$$\frac{\delta \bar{E}_2}{\delta \epsilon} \Big|_{\epsilon=0} = \iint_A \frac{\phi'(\|\nabla J\|)}{\|\nabla J\|} \langle \nabla J, \nabla \delta J \rangle dA$$

\Rightarrow int. by parts

need to apply some special theorem here.

$$\cancel{\int \phi'(\|\nabla J\|) \cdot \nabla J}$$

variables

$$\Rightarrow \langle \nabla J, \nabla \rangle = 0$$

$$= \int_A \frac{\phi'(\|\nabla J\|)}{\|\nabla J\|} \langle \nabla J, \cancel{\nabla \delta J} \rangle dA - \int_A \nabla \left(\frac{\phi'(\|\nabla J\|)}{\|\nabla J\|} \right) \delta J dA$$

Total solution is a loop:

initial guess for current iteration, $J_{in, out}$

then solve for $J_{in, out}$, then update curve

solve for $J_{in, out}$, then update curve

repeat to convergence.

EZ7311: initialize system

initial guess for current iteration, $J_{in, out}$

① solve for $J_{in, out}$

$$J_{in} - \Delta J_{in} = I \quad \omega / \langle \nabla J_{in}, N \rangle = 0$$

$$J_{out} - \Delta J_{out} = I \quad \omega / \langle \nabla J_{out}, N \rangle = 0$$

will usually require some sort of iterative solver.

② update curve according to gradient descent

$$\frac{dC}{dt} = [(I - J_{in})^2 - (I - J_{out})^2]N + [\|\nabla J_{in}\|^2 - \|\nabla J_{out}\|^2]N$$

THE BASIC IDEA IS: find the point where $\frac{dC}{dt} = 0$ (minimum)

$$+ KN$$

MINIMUM OF THE FUNCTION C IS FOUND BY SOLVING THE EQUATION $\frac{dC}{dt} = 0$

③ repeat to convergence.

initial guess for current iteration, $J_{in, out}$

then solve for $J_{in, out}$, then update curve

repeat to convergence.

initial guess for current iteration, $J_{in, out}$

then solve for $J_{in, out}$, then update curve

repeat to convergence.

$$x(t) = \exp(-t^2/2\sigma_0^2)$$

Last final struggle in Mumford-Shah segmentation

→ last time we had two parts to the energy minimization procedure.

1 compute minimizing J .

2 evolve/update curve using gradient descent.



TODAY: how to implement curve evolution (sort of).

Assumption premise: only one curve, no merging / splitting.

- * can define region through parametric representation of boundary curve.
- * will try to use Matlab spline toolbox.

what's that?

a popular curve/surface interpolation technique

3.3.3 Глобальное сглаживание изображения при помощи

ноябрь

использование для сглаживания изображения. Для этого вначале

мы будем использовать методы, которые в принципе не являются

лучшими способами. Эти методы очень эффективны для техни

ких изображений, которые имеют небольшое количество пикселей

и высокую яркость. А если же мы имеем изображение с

HAVE TO IMPLEMENT CURVE EVOLUTION

numerical methods for solving differential equations for geometric problems [121]

numerical methods for solving differential equations for geometric problems [121]

- if there is one curve that does not split nor merge,
then a parametric spline representation can be used.

numerical methods for solving differential equations for geometric problems [121]

MATLAB has a spline toolbox

numerical methods for solving differential equations for geometric problems [121]

a spline is a polynomial approximation to a curve.

if defined properly, it is smooth.

numerical methods for solving differential equations for geometric problems [121]

idea behind spline: cubic spline

numerical methods for solving differential equations for geometric problems [121]

$$P(x) = a_3x^3 + a_2x^2 + a_1x + a_0 = \begin{bmatrix} x^3 & x^2 & x & 1 \end{bmatrix} \begin{Bmatrix} a_3 \\ a_2 \\ a_1 \\ a_0 \end{Bmatrix}$$

numerical methods for solving differential equations for geometric problems [121]

if have $P(0) = y_0$, $P'(0) = v_0$, $P(1) = y_1$, $P'(1) = v_1$,

numerical methods for solving differential equations for geometric problems [121]

then polynomial is completely determined

$$\begin{Bmatrix} y_0 \\ v_0 \\ y_1 \\ v_1 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 3 & 2 & 1 & 0 \end{bmatrix} \begin{Bmatrix} a_3 \\ a_2 \\ a_1 \\ a_0 \end{Bmatrix}$$

numerical methods for solving differential equations for geometric problems [121]

numerical methods for solving differential equations for geometric problems [121]

so, in MATLAB we can take points on curve

and do

points on curve for each parameter value

$\Rightarrow \text{mySp} = \text{spline}(p, X)$

bp



points on curve for each $p[i]$

~~for~~ X is a matrix.

$$p[i] \in [0, 1]$$

more material covered in class

to evaluate, do

conveniently to get job done, use ppval function to evaluate the spline

\Rightarrow curve pts = $\text{ppval}(\text{mySp}, p)$



collection of parameter values

do this with new parameter values

selection of tool appropriate for task

more convenient and easier to use

to get derivatives

directional derivative

$\text{spDer} = \text{fnder}(\text{mySp})$

fnder

$\text{fnder}(\text{mySp}, k)$

order of derivative

to evaluate a spline can also use

fnval

function to evaluate a spline at a point

processor friendly interface for problems requiring many evaluations

to get normal vector direction, just rotate tangent vector.

example:

eg classmate ~~closed~~

~~c = spline(p, X)~~

closed

a spline connects a bunch of polynomials smoothly

\Rightarrow e.g. matching derivatives

given ~~and~~ points on curve

more often, derivatives are important because it's smooth

$$P_1(x) = a_3 x^3 + a_2 x^2 + a_1 x + a_0$$

$$P_2(x) = b_3 x^3 + b_2 x^2 + b_1 x + b_0$$

want

$$P_1(0) = y_0, \quad P_1(1) = y_1$$

$$P_2(0) = y_1, \quad P_2(1) = y_2$$

so we want to make sure that the first derivative is continuous at the endpoints [42]

continuous spline [43] connects two curves of degree n so that the derivatives at the endpoints are continuous

only 4 constraints, what about other 4?

so for example if we want to make sure that the first derivative is continuous at the endpoints [40, 118, 145]

$$\dot{P}_1(1) = \dot{P}_2(0)$$

adds two more

endpoints are free to vary in velocity or acceleration.

① can be fixed by defining vel. or acc.

② if endpoints match (closed curve), then matching derivatives fully constrain system.

$C = \text{csape}(p, \text{points}, \text{'periodic});$

$C-p = \text{fnnder}(C, 1);$

$C-pp = \text{fnnder}(C, 2);$

↑ multiple intervals off at first

↳ this gives splines.

To get actual values, need to evaluate

$\text{curve_pp} = \text{fnval}(C-p, p)$

↳ list of ~~points~~ parameter values
to evaluate at.

best to parametrize by arc-length

: .
: .
: .

first get arclength

$$s = \int_0^1 \|C_p\| dp = \sum_{i=2}^k \|C(i) - C(i-1)\| \cdot (p(i) - p(i-1))$$

each term is locally ds

want ~~constant~~ ds , or near ~~constant~~
unit unity

$C = \text{csape}($