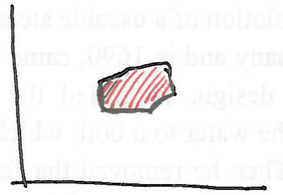
Curves and Curve Evolution:

Observation: the boundary of an object forms a closed curve.



part of Mumford-Shah energy functional seeks to minimize the length of the boundary

$\Rightarrow$

minimize length of closed curve.

* this is a problem in curve evolution theory.

to do this will require some preliminaries.

* we have a closed curve, defined by $C: S^1 \to \mathbb{R}^2$

(takes the circle and maps it to curve in $\mathbb{R}^2$)

alternatively $C: [0,1] \to \mathbb{R}^2$ some parametization

$C: [0,L] \to \mathbb{R}^2$ arc-length parametization.

L length of curve

uses Euclidean 2-norm, $s = \int_0^p \langle C_p, C_p \rangle^{\frac{1}{2}} dp$

~~wanted~~ length of curve is $L = \int_0^1 \langle C_p, C_p \rangle^{1/2} dp = \int_0^1 \|C_p\| dp$

or $L = \int_0^L ds$

$\Rightarrow$ $ds = \|C_p\| dp$

note also that $\frac{\partial s}{\partial p} = \|C_p\|^{1/2}$

$\Rightarrow$

$$\frac{1}{\|C_p\|^{1/2}} \frac{\partial}{\partial p} = \frac{\partial}{\partial s}$$

$$\frac{\partial}{\partial p} f(s) = f' \frac{\partial s}{\partial p} = \|C_p\|^{1/2} f'(s) \Rightarrow \frac{1}{\|C_p\|^{1/2}} \frac{\partial}{\partial p} f(s) = \frac{\partial}{\partial s} f(s)$$

$$\frac{\partial}{\partial s} f(s) = f'(s)$$

can relate their differentials & area forms

Other properties, definitions

$$T = \frac{\partial C}{\partial s} = \frac{1}{\|C_p\|} \frac{\partial C}{\partial p} \quad C_s \quad \frac{1}{\|C_p\|} C_p$$

$\langle T, T \rangle = 1$

$\Rightarrow$ $\langle T, \frac{\partial T}{\partial s} \rangle = 0$

$\Rightarrow$ $T \perp \frac{\partial T}{\partial s}$

$$N = \frac{\partial T / \partial s}{\|\partial T / \partial s\|}$$

well, $\|\frac{\partial T}{\partial s}\| = K$

called curvature

$$\frac{\partial T}{\partial s} = KN$$

$$\frac{\partial N}{\partial s} = ?? \quad KT + ?N$$

$$L(C+\epsilon V) = \int_0^1 \|C_p+\epsilon V_p\|\,dp$$

$\Rightarrow$

$$\frac{\partial L}{\partial \epsilon}\Big|_{\epsilon=0} = \int_0^1 \frac{1}{2}\cdot 2\cdot \frac{\langle C_p+\epsilon V_p, V_p\rangle}{\langle C_p+\epsilon V_p, C_p+\epsilon V_p\rangle^{1/2}}\Big|_{\epsilon=0} dp$$

$\|C_p\|dp = ds$

$\frac{\partial}{\partial s} = \frac{1}{\|C_p\|}\frac{\partial}{\partial p}$

$$= \int_0^1 \frac{\langle C_p, V_p\rangle}{\langle C_p, C_p\rangle^{1/2}}$$

$\frac{\partial}{\partial s} T = KN$

$$= \int_0^1 \frac{1}{\|C_p\|}\langle C_p, V_p\rangle\, dp$$

$$= \cancel{\left\langle \frac{C_p}{\|C_p\|}, V\right\rangle\Big|_0^1} - \int_0^1 \left\langle \frac{d}{dp}\frac{C_p}{\|C_p\|}, V\right\rangle dp$$

$$= -\int_0^L \cancel{\frac{\|C_p\|}{dp}}\,\|C_p\|\left\langle \frac{1}{\|C_p\|}\frac{\partial}{\partial p}\frac{C_p}{\|C_p\|}, V\right\rangle dp$$

$$= -\int_0^L \left\langle \frac{\partial}{\partial s} C_s, V\right\rangle \cancel{ds}\,\|C_p\|\,dp$$

$$= -\int_0^L \langle C_{ss}, V\rangle\, ds$$

$$\frac{\partial L}{\partial \epsilon}\Big|_{\epsilon=0} = -\int_0^L \langle KN, V\rangle\, ds$$

$C_{ss} = \frac{\partial}{\partial s}\frac{\partial}{\partial s} C$

$= \frac{1}{\|C_p\|}\frac{\partial}{\partial p}\cdot\frac{1}{\|C_p\|}\frac{\partial}{\partial p} C$

$\Rightarrow \quad V = KN$

$= \frac{1}{\|C_p\|}\frac{\partial}{\partial p}\frac{1}{\|C_p\|} C_p$

$= \frac{1}{\|C_p\|^2} C_{pp} - \frac{1}{\|C_p\|}\frac{C_p\langle C_p, C_{pp}\rangle}{\|C_p\|^3}$

$\Rightarrow$ to shrink curve, $\frac{dC}{dt} = KN \qquad C_t = KN$

$= \frac{C_{pp}}{\|C_p\|^2} - \frac{C_p\langle C_p, C_{pp}\rangle}{\|C_p\|^4}$

arc-length

$$\mathcal{L} = \int_0^1 \| C_p \| \, dp \quad = \quad \int_0^L ds \qquad\qquad ds = \frac{1}{\|C_p\|} \, dp$$

$$\mathcal{L} = \int_0^1 \langle C_p, C_p \rangle^{1/2} \, dp \qquad \longleftarrow \text{Euclidean 2-norm}$$

$$\mathcal{L}(C_p + \sigma \delta C) = \int_0^1 \langle C_p + \sigma\delta C_p, C_p + \sigma\delta C_p \rangle^{1/2} \, dp$$

$\Rightarrow$

$$\frac{\delta \mathcal{L}}{\delta \sigma}\Big|_{\sigma=0} = \int_0^1 \frac{1}{2} \cdot 2 \cdot \frac{\langle \delta C_p, C_p + \sigma \delta C_p \rangle}{\langle C_p + \sigma\delta C_p, C_p + \sigma \delta C_p \rangle} \Big|_{\sigma = 0} \, dp$$

$$\frac{\partial T}{\partial s} = -\kappa N$$

$$= \int_0^1 \frac{\langle C_p, \delta C_p \rangle}{\langle C_p, C_p \rangle} \, dp$$

$$= \int_0^1 \left\langle \frac{C_p}{\|C_p\|} ; \delta C_p \right\rangle dp$$

$$\frac{d}{dp} \langle C_p, C_p \rangle^{-1/2} = -\frac{1}{2} \langle C_p, C_p \rangle^{-3/2} \cdot 2 \langle C_p, C_{pp} \rangle$$

$$= \cancel{\frac{C_p}{\|C_p\|} \cdot \delta C \Big|_0^1} - \int_0^1 \left\langle \frac{d}{dp} \frac{C_p}{\|C_p\|}, \delta C \right\rangle dp$$

$$= -\int_0^1 \left\langle \frac{C_{pp}}{\|C_p\|} - \frac{\langle C_p, C_{pp} \rangle C_p}{\|C_p\|^3}, \delta C \right\rangle dp$$

Suppose curve is a circle:

$$C(s) = (r\cos(s/r),\ r\sin(s/r))$$

$\Rightarrow$

$$T = \frac{\partial C}{\partial s} = -\sin\left(\frac{s}{r}\right)\frac{\partial}{\partial x} + \cos\left(\frac{s}{r}\right)\frac{\partial}{\partial y}$$

$$= \left(-\sin\left(\frac{s}{r}\right),\ \cos\left(\frac{s}{r}\right)\right)$$

As a vector!

$$\kappa N = \frac{\partial T}{\partial s} = \left(-\frac{1}{r}\cos\left(\frac{s}{r}\right),\ -\frac{1}{r}\sin\left(\frac{s}{r}\right)\right) = \frac{1}{r}\left(-\cos\left(\frac{s}{r}\right),\ \sin\left(\frac{s}{r}\right)\right)$$

$$= \frac{1}{r} K$$

$\Rightarrow$

$$\kappa = \frac{1}{r}$$ Curvature is constant

~~descri~~ inversely proportional to radius of circle

$\Rightarrow$ curvature describes "amount of bending" of curve at a point. It has local dependence on the curve (through derivatives)

What about minimizing area?

$$L = \int_R dA$$

from div. theorem, $\displaystyle\int_{\partial R} F\cdot N\,ds = \int_R \nabla\cdot F\,dA = \int_R f\,dA.$

so, define

$$\mathcal{E}(C) = \int_C F\cdot N\,ds$$

but for planar curves, $N = JT$ where $J = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}$

$\Rightarrow$

$$\mathcal{E}(C) = \int_C F\cdot J\overset{T}{\cancel{}}\,ds = \int_{\cancel{C}0}^{1} F\cdot J\frac{C_p}{\|C_p\|}\|C_p\|\,dp = \int_0^1 F\cdot JC_p\,dp$$

$$\mathcal{E}(C+\epsilon V) = \int_{0\,\cancel{C+\epsilon V}}^{1} \overset{(C+\epsilon V)}{F}\cdot J(C_p+\epsilon V_p)\,dp$$

$$\left.\frac{\delta\mathcal{E}}{\delta\epsilon}\right|_{\epsilon=0} = \int_0^1 \left.\Big[(DF\cdot V)\cdot J(C_p+\epsilon V_p) + F\cdot JV_p\Big]\right|_{\epsilon=0} dp$$

$$= \int_0^1 \big[(DF\cdot V)\cdot JC_p + F\cdot JV_p\big]\,dp$$

$$= \cancel{\int_0 F\cdot JV\Big|_0^1} + \int_0^1 \Big[(DF\cdot V)\cdot JC_p - \frac{\partial}{\partial p}(F)\cdot JV\Big]\,dp$$

$$= \int_0^1 \big[(DF\cdot V)\cdot JC_p - (DF\cdot C_p)\cdot JV\big]\,dp$$

$$= \int_0^1 \big[(JC_p)^T DF\cdot V - (DF\cdot C_p)\cdot JV\big]\,dp$$

$$= \int_0^1 \left[(J C_p)^T DF - (DF \cdot C_p)^T J \right] V \, dp$$

$$= \int_0^1 \left[C_p^T (J^T DF - DF^T J) \right] V \, dp$$

$$\left[\begin{smallmatrix} & 1 \\ -1 & 0 \end{smallmatrix} \right]$$

define $\quad \phi J \equiv J^T DF - DF^T J$

$$\Rightarrow \quad \phi = -J^T DF J - DF$$

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial F^1}{\partial x^1} & \frac{\partial F^1}{\partial x^2} \\ \frac{\partial F^2}{\partial x^1} & \frac{\partial F^2}{\partial x^2} \end{bmatrix}$$

$$\begin{bmatrix} \frac{\partial F^2}{\partial x^1} & \frac{\partial F^2}{\partial x^2} \\ \frac{\partial F^1}{\partial x^1} & -\frac{\partial F^1}{\partial x^2} \end{bmatrix} - \begin{bmatrix} -\frac{\partial F^2}{\partial x^2} & \frac{\partial F^1}{\partial x^1} \\ \frac{\partial F^2}{\partial x^2} & \frac{\partial F^2}{\partial x^1} \end{bmatrix}$$

$$= \int_0^1 C_p^T \phi J V \, dp$$

$$= \int_0^1 \phi J^T C_p V \, dp$$

$$= \int_0^1 \phi J^T \frac{C_p}{\|C_p\|} V \, \|C_p\| dp$$

$$= \int_0^L \phi J^T C_s V \, ds$$

$$= \int_0^L -\phi N V \, ds$$

$$= \int_0^L -\phi \langle N, V \rangle \, ds$$

$$= \int_0^L \langle -\phi N, V \rangle \, ds$$

$$\Rightarrow$$

$$\int_0^L \langle (\nabla \cdot F) N, V \rangle \, ds$$

$$= \int_0^L \langle f N, V \rangle \, ds$$

$$= \int_C \langle f N, V \rangle \, ds$$

$$\cancel{\phi = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}}$$

$$\phi = -\begin{bmatrix} \frac{\partial F^1}{\partial x^1} & \frac{\partial F^2}{\partial x^1} \\ \frac{\partial F^1}{\partial x^2} & \frac{\partial F^2}{\partial x^2} \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial F^1}{\partial x^1} & \frac{\partial F^1}{\partial x^2} \\ \frac{\partial F^2}{\partial x^1} & \frac{\partial F^2}{\partial x^2} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} -\frac{\partial F^1}{\partial x^2} & \frac{\partial F^1}{\partial x^1} \\ -\frac{\partial F^2}{\partial x^2} & \frac{\partial F^2}{\partial x^1} \end{bmatrix}$$

$$+ \begin{bmatrix} -\frac{\partial F^2}{\partial x^2} & \frac{\partial F^2}{\partial x^1} \\ +\frac{\partial F^1}{\partial x^2} & -\frac{\partial F^1}{\partial x^1} \end{bmatrix}$$

$$\Rightarrow$$

$$\phi = \begin{bmatrix} -\frac{\partial F^1}{\partial x^1} - \frac{\partial F^2}{\partial x^2} & 0 \\ 0 & -\frac{\partial F^1}{\partial x^1} - \frac{\partial F^2}{\partial x^2} \end{bmatrix}$$

$$\Rightarrow$$

$$\phi = -\nabla \cdot F$$

but, what does this flow have anything to do with the Energy functional?

well, $\iint_D (I-J)^2\,dA$ is one of the terms in the functional.

how do we deal w/ it? $\rightarrow$ start w/ assumption of one region

$\Rightarrow$

$$\iint_R (I-J_{in})^2\,dA + \iint_{D\setminus R} (I-J_{out})^2\,dA \qquad J = \begin{cases} J_{in} & x \in R \\ J_{out} & x \notin R \end{cases}$$

same boundaries, but different inward pointing normals.

$\Rightarrow$

$$\frac{\partial C}{\partial t} = \left[(I-J_{in})^2 - (I-J_{out})^2\right] \underbrace{N}_{\text{inward pointing normal?}}$$

so, we can handle this if J is fixed.

what about $\iint \phi(\|\nabla J\|)\,dA$? same deal

$$\frac{\partial C}{\partial t} = \left[\phi(\|\nabla J_{in}\|) - \phi(\|\nabla J_{out}\|)\right] N$$

so, we know how to deal w/ the Γ part.

now, what about the J part?

need to look at variations w/respect to J.

$$\mathcal{E}(J,\Gamma) = \iint_D (I-J)^2\,dA + \iint_{D\setminus\Gamma} \phi(\|\nabla J\|)\,dA + \nu \underbrace{\int_\Gamma ds}_{\text{does not depend on } J}$$

$|\Gamma|$

does not depend on J

$\Rightarrow$ ignore for now.

consider image part only, with Γ fixed.

$$\bar{\mathcal{E}}(J,\Gamma) = \underbrace{\iint_D (I-J)^2\,dA}_{\text{do this first}} + \iint_{D\setminus\Gamma} \phi(\|\nabla J\|)\,dA$$

suppose, we have a linear function of I, $f(I+\epsilon\,\delta I) = f(I) + f(\delta I)$

$\Rightarrow$

$f(I) = \begin{cases} \text{constant} \\ \cancel{\text{something}} \end{cases}$

$$\bar{\mathcal{E}}_1(J+\epsilon\,\delta J,\Gamma) = \iint_D f^2(I-J-\epsilon\,\delta J)\,dA$$

$$\left.\frac{\delta \bar{\mathcal{E}}_1}{\delta \epsilon}\right|_{\epsilon=0} = -\iint_D 2f(I-J-\epsilon\,\delta J)\cdot f'(I-J-\epsilon\,\delta J)\cdot \delta J \Big|_{\epsilon=0} dA$$

$$= -\iint_D \underbrace{\big(2f(I-J)\cdot f'(I-J)\big)}_{\text{this is minimizing solution}}\cdot \delta J\,dA$$

$$2f(I-J)\cdot f'(I-J) = 0$$

for our case

$$2(I-J) = 0$$

$$\Rightarrow J = I$$

what's the deal here? ~~Nothing~~, This part is correct, but we forgot about the second term

$\phi'(x^2) = 2x$

$\Rightarrow$

$$\left.\frac{\delta \bar{\xi}_2}{\delta \epsilon}\right|_{\epsilon=0} = -\int_A \nabla\left[\frac{\phi'(\|\nabla J\|)}{\|\nabla J\|} \nabla J\right] \delta u \, dA$$

MISSING STEPS HERE!!!

$\Rightarrow$

$$\left.\frac{\delta \xi}{\delta \epsilon}\right|_{\epsilon=0} = \iint_R \left[-2(I-J_{in}) - 2\nabla\left[\frac{\phi'(\|\nabla J_{in}\|)}{\|\nabla J_{in}\|} \nabla J_{in}\right]\right] \delta u \, dA$$

$$+ \iint_{D\setminus R} \left[-2(I-J)_{out} - 2\nabla\left[\frac{\phi'(\|\nabla J_{out}\|)}{\|\nabla J_{out}\|} \nabla J_{out}\right]\right] \delta u \, dA$$

$\Rightarrow$ get critical point:

$$-(I - J_{in}) - \nabla\left[\frac{\phi'(\|\nabla J_{in}\|)}{\|\nabla J_{in}\|} \nabla J_{in}\right] = 0$$

$$-(I - J_{out}) - \nabla\left[\frac{\phi'(\|\nabla J_{out}\|)}{\|\nabla J_{out}\|} \nabla J_{out}\right] = 0$$

classical definition of ϕ is $\phi(\alpha) = \alpha^2 \Rightarrow \phi' = 2\alpha$

$\Rightarrow$

$$-I + J_{in} - \nabla[\nabla J_{in}] = 0 \quad \text{w/ } \langle \nabla J_{in}, N\rangle = 0$$

$\Rightarrow$

$$-I + J_{in} - \Delta J_{in} = 0 \quad \text{w/ } \langle \nabla J_{in}, N\rangle = 0$$

let's check out the second term. (we may need to specify further!)

$$\iint_{D\setminus\Gamma} \phi(\|\nabla J\|)\,dA = \iint_{R} \phi(\|\nabla J_{in}\|)\,dA + \iint_{D\setminus R} \phi(\|\nabla J_{out}\|)\,dA .$$

ok, so important thing is to get $\iint_{A} \phi(\|\nabla J\|)\,dA$

$$\frac{\delta \bar{E}_2}{\delta \epsilon}\Big|_{\epsilon=0} \qquad \bar{E}_2(J+\epsilon\,\delta J, \Gamma) = \iint_{A} \phi(\|\nabla J + \epsilon \nabla \delta J\|)\,dA$$

$$\frac{\delta \bar{E}_2}{\delta \epsilon}\Big|_{\epsilon=0} = \iint_{A} \phi'(\|\nabla J + \epsilon \nabla \delta J\|) \cdot \frac{\langle \nabla \delta J, \nabla J + \epsilon \nabla \delta J\rangle}{\|\nabla J + \epsilon\, \delta \nabla J\|}\Big|_{\epsilon=0} dA$$

$\Rightarrow$

$$\frac{\delta \bar{E}_2}{\delta \epsilon}\Big|_{\epsilon=0} = \iint_{A} \phi'(\|\nabla J \cancel{+ \ldots}\|) \cdot \frac{\langle \nabla \delta J, \nabla J\rangle}{\|\nabla J \cancel{+ \ldots}\|}\, dA$$

$\Rightarrow$

$$\frac{\delta \bar{E}_2}{\delta \epsilon}\Big|_{\epsilon=0} = \iint_{A} \frac{\phi'(\|\nabla J\|)}{\|\nabla J\|} \langle \nabla J, \nabla \delta J \rangle\, dA$$

$\Rightarrow$ int. by parts

need to apply some special theorem here.

$$\cancel{= \frac{\phi'(\|\nabla J\|)}{\|\nabla J\|} \cdot \delta J \Big|_{\partial A}} \qquad \langle \nabla J, N \rangle = 0$$

$$= \int_{\partial A} \frac{\phi'(\|\nabla J\|)}{\|\nabla J\|} \underbrace{\langle \nabla J, \cancel{N}\,\delta u \rangle}_{\text{vanishes}}\, ds - \int_{A} \nabla\!\left(\frac{\phi'(\|\nabla J\|)}{\|\nabla J\|} \nabla J \right) \delta u\, dA$$

Total solution is a doozy:

solve for J_{in}, J_{out}, then update curve

repeat to convergence.

⓪ initialize system

① solve for J_{in}, J_{out}

$$J_{in} - \Delta J_{in} = I \qquad \text{w/ } \langle \nabla J_{in}, N \rangle = 0$$

$$J_{out} - \Delta J_{out} = I \qquad \text{w/ } \langle \nabla J_{out}, N \rangle = 0$$

· will usually require some sort of iterative solver.

② update curve according to gradient descent

$$\frac{\partial C}{\partial t} = \left[(I - J_{in})^2 - (I - J_{out})^2 \right] N + \left[\|\nabla J_{in}\|^2 - \|\nabla J_{out}\|^2 \right] N$$

$$+ \kappa N$$

③ repeat to convergence.

Last final struggle in Mumford-Shah segmentation

→ last time we had two parts to the energy minimization procedure.

$\boxed{1}$ compute minimizing J.

$\boxed{2}$ evolve/update curve using gradient descent.

TODAY: how to implement curve evolution (sort of).

Assumption/premise: only one curve, no merging/splitting.

* can define region through parametric representation of boundary curve.
* will try to use Matlab spline toolbox.

what's that?

a popular curve/surface interpolation technique

HAVE TO IMPLEMENT CURVE EVOLUTION

- if there is one curve that does not split nor merge, then a parametric spline representation can be used.

MATLAB has a spline toolbox

a spline is a polynomial approximation to a curve.

if defined properly, it is smooth.

idea behind spline. cubic spline

$$P(x) = a_3x^3 + a_2x^2 + a_1x + a_0 = \begin{bmatrix} x^3 & x^2 & x & 1 \end{bmatrix} \begin{Bmatrix} a_3 \\ a_2 \\ a_1 \\ a_0 \end{Bmatrix}$$

if have $P(0) = y_0$, $\dot{p}(0) = v_0$, $P(1) = y_1$, $\dot{p}(1) = v_1$

then polynomial is completely determined

$$\begin{Bmatrix} y_0 \\ v_0 \\ y_1 \\ v_1 \end{Bmatrix} = \begin{bmatrix} \cancel{x^3} & \cancel{x^2} & \cancel{x} & 1 \\ \cancel{3x^2} & \cancel{2x} & \cancel{1} & 0 \\ & & & \\ & & & \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 3 & 2 & 1 & 0 \end{bmatrix} \begin{Bmatrix} a_3 \\ a_2 \\ a_1 \\ a_0 \end{Bmatrix}$$

so, in MATLAB can take points on curve and do

mysp = spline(p, X)

points on curve for each p[i]

X is a matrix

~~p ∈ [0,1]~~

p[i] ∈ [0,1]

to evaluate, do

curvepts = ppval(mysp, p)

collection of parameter values

resulting points

to get derivatives

spdot = fnder(mysp)

fnder(mysp, k)

order of derivative

directional derivative

fndir

to evaluate a spline can also use

fnval

to get normal vector direction, just rotate tangent vector.

example: ~~c = spline(p, X)~~

closed.

a spline connects a bunch of polynomials smoothly

$\Rightarrow$ e.g. matching derivatives

given points on curve

$$p_1(x) = a_3x^3 + a_2x^2 + a_1x + a_0$$

$$p_2(x) = b_3x^3 + b_2x^2 + b_1x + b_0$$

want

$$p_1(0) = y_0, \quad p_1(1) = y_1$$

$$p_2(0) = y_1, \quad p_2(1) = y_2$$

only 4 constraints, what about other 4?

well,

$$\dot{p}_1(1) = \dot{p}_2(0)$$

$$\ddot{p}_1(1) = \ddot{p}_2(0)$$

adds two more

endpoints are free to vary in velocity or acceleration.

(1) can fixed by defining vel. or acc.

(2) If endpoints match (closed curve), then matching derivatives fully constrain system.

```
C = csape(p, points, 'periodic');

C_p = fnder(C, 1);

C_pp = fnder(C, 2);
```

this gives splines.

To get actual values, need to evaluate

```
curve_pp = fnval(C_p, p)
```

list of ~~points~~ parameter values to evaluate at.

---

best to parametrize by arc-length

first get arclength

$$\ell = \int_0^1 \|C_p\| \, dp = \underbrace{\sum_{i=2}^{k} \|C(i)-C(i-1)\| \cdot (p(i)-p(i-1))}_{\text{each term is locally } ds}$$

want ~~constant~~ unit ds, or near ~~constant~~ unity

C = csape(