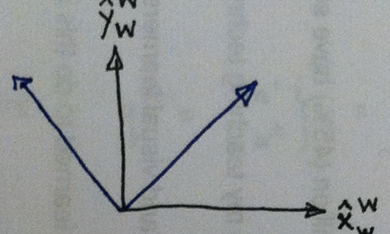
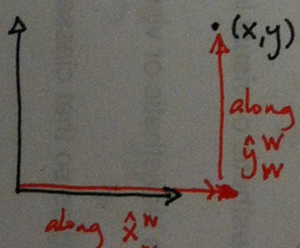
CHANGE OF COORDINATES : ROTATION ONLY.

- world frame is what it is: W.
- suppose now that the camera is rotated by θ relative to the world frame. the camera frame will be given by C.



we denote by $\hat{x}^W_W / \hat{y}^W_W$ the x-axis / y-axis unit vector of W in frame W.

a coordinate vector $\begin{Bmatrix} x \\ y \end{Bmatrix}^W$ in the world frame really means to move by x along the $\hat{x}^W_W$ vector, then by y along the $\hat{y}^W_W$ vector.



$$q^W = \begin{Bmatrix} x \\ y \end{Bmatrix}^W = x \cdot \hat{x}^W_W + y\, \hat{y}^W_W$$

also sometimes denoted by

$$\vec{\alpha}^W = \begin{Bmatrix} \alpha^1 \\ \alpha^2 \end{Bmatrix}^W = \alpha^1 \cdot \hat{x}^W_W + \alpha^2\, \hat{y}^W_W$$

or even $q^W = \begin{Bmatrix} q^1 \\ q^2 \end{Bmatrix}^W = \ldots$

or anything similar . . .

we can change coordinates by switching the $\hat{x}$ & $\hat{y}$ vectors around. or better put by figuring out what α^1 & α^2 are given different choices of $\hat{x}$ & $\hat{y}$.

so, given q^W what is q^C?

likewise, given q^C what is q^W?

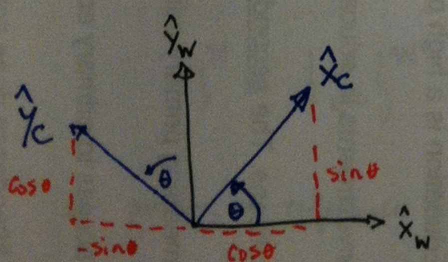
since the answer to one is the inverse of the others, we just need to figure out one of the answers.

let $$q^W = \begin{Bmatrix} \alpha^1 \\ \alpha^2 \end{Bmatrix}^W = \alpha^1 \cdot \hat{x}_W^W + \alpha^2 \hat{y}_W^W .$$

we also have $\hat{x}_C^C$ & $\hat{y}_C^C$. but to change coordinates, we need the mixed version $\hat{x}_C^W$ & $\hat{y}_C^W$ (or the flip)

$\hat{x}_C^W \rightarrow$ ~~x-axis thg aligned unit vector~~
unit vector aligned with x-axis of frame C given in frame W

$\hat{y}_C^W \rightarrow$ unit vector aligned with y-axis of frame C given in frame W.



frame C is rotated by θ relative to frame W

Now,

$$\hat{x}_C^W = \begin{Bmatrix} \cos\theta \\ \sin\theta \end{Bmatrix}^W = \cos\theta \, \hat{x}_W^W + \sin\theta \, \hat{y}_W^W$$

$$\hat{y}_C^W = \begin{Bmatrix} -\sin\theta \\ \cos\theta \end{Bmatrix}^W = -\sin\theta \, \hat{x}_W^W + \cos\theta \, \hat{y}_W^W$$

$\Rightarrow$

$$q^{c} = \begin{Bmatrix} \beta^1 \\ \beta^2 \end{Bmatrix}^{c} = \beta^1 \hat{x}_c^{c} + \beta^2 \hat{y}_c^{c}$$

$\Rightarrow$ move to world coordinates descriptions for $\hat{x}_c$ & $\hat{y}_c$

$$q^{W} = \begin{Bmatrix} \alpha^1 \\ \alpha^2 \end{Bmatrix}^{W} = \beta^1 \hat{x}_c^{W} + \beta^2 \hat{y}_c^{W}$$

$$= \beta^1 \left[\cos\theta \, \underline{\hat{x}_W^W} + \sin\theta \, \underline{\hat{y}_W^W} \right] + \beta^2 \left[-\sin\theta \, \underline{\hat{x}_W^W} + \cos\theta \, \underline{\hat{y}_W^W} \right]$$

$\Rightarrow$ Consolidate the $\hat{x}_W^W$ & $\hat{y}_W^W$ terms

$$q^{W} = \begin{Bmatrix} \alpha^1 \\ \alpha^2 \end{Bmatrix} = \left[\beta^1 \cos\theta - \beta^2 \sin\theta \right] \hat{x}_W^W + \left[+\beta^1 \sin\theta + \beta^2 \cos\theta \right] \hat{y}_W^W$$

$\Rightarrow$

$$q^{W} = \begin{Bmatrix} \alpha^1 \\ \alpha^2 \end{Bmatrix}^{W} = \begin{Bmatrix} \beta^1 \cos\theta - \beta^2 \sin\theta \\ \beta^1 \sin\theta + \beta^2 \cos\theta \end{Bmatrix}^{W}$$

↑ note that this side is in terms of the coordinate values from q^c multiplied by $\sin\theta$ or $\cos\theta$.

we can factor out the β^1 & β^2 terms

$\Rightarrow$ factor out $\begin{Bmatrix} \beta^1 \\ \beta^2 \end{Bmatrix}$

$$q^{W} = \begin{Bmatrix} \alpha^1 \\ \alpha^2 \end{Bmatrix}^{W} = \left(\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{Bmatrix} \beta^1 \\ \beta^2 \end{Bmatrix} \right)^{W}$$

$\Rightarrow$ putting in reference frames

$$q^{W} = \begin{Bmatrix} \alpha^1 \\ \alpha^2 \end{Bmatrix}^{W} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}_{c}^{W} \begin{Bmatrix} \beta^1 \\ \beta^2 \end{Bmatrix}^{c} = R_c^W q^c$$

where

$$R_c^W = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

so,

$$q^W = R_c^W q^c$$

further note that

$$R_c^W = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}_c^W = \begin{bmatrix} \hat{x}_c^W & \hat{y}_c^W \end{bmatrix}$$

so, basically, the rotation matrix consists of the unit axis vectors of the old frame (here c) written with respect to the new frame (here W). That makes sense!

The inverse relationship is

$$q^c = R_W^c q^W$$

which means that R_c^W and R_W^c are inverses.

$$R_c^W = (R_W^c)^{-1} \qquad \text{AND} \qquad R_W^c = (R_c^W)^{-1}$$

also makes sense. inverting the matrix is like inverting the change of coordinates.

In the 3D case

$$R_c^W = \begin{bmatrix} | & | & | \\ \hat{x}_c^W & \hat{y}_c^W & \hat{z}_c^W \\ | & | & | \end{bmatrix}$$ ← 3x3 matrix

where $\hat{x}_c^W, \hat{y}_c^W, \hat{z}_c^W$ are 3D vectors.

remember that they are unit vectors & ~~basis~~ vectors, so...

$$(R_c^W)^T = \begin{bmatrix} -(\hat{x}_c^W)^T- \\ -(\hat{y}_c^W)^T- \\ -(\hat{z}_c^W)^T- \end{bmatrix}$$ ← 3x3 matrix

⇒

$$(R_c^W)^T \cdot R_c^W = \begin{bmatrix} -(\hat{x}_c^W)^T- \\ -(\hat{y}_c^W)^T- \\ -(\hat{z}_c^W)^T- \end{bmatrix} \begin{bmatrix} | & | & | \\ \hat{x}_c^W & \hat{y}_c^W & \hat{z}_c^W \\ | & | & | \end{bmatrix}$$

$$= \left[\begin{array}{c|c|c} (\hat{x}_c^W)^T \hat{x}_c^W & (\hat{x}_c^W)^T \hat{y}_c^W & (\hat{x}_c^W)^T \hat{z}_c^W \\ \hline (\hat{y}_c^W)^T \hat{x}_c^W & (\hat{y}_c^W)^T \hat{y}_c^W & (\hat{y}_c^W)^T \hat{z}_c^W \\ \hline (\hat{z}_c^W) \hat{x}_c^W & (\hat{z}_c^W) \hat{y}_c^W & (\hat{z}_c^W) \hat{z}_c^W \end{array}\right]$$ ← 3x3 matrix

⇒ the basis vectors are orthogonal to each other

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Holy cow!

$$(R^W_c)^T R^W_c = \mathbb{1} \qquad \text{identity matrix}$$

But ...

$$(R^W_c)^{-1} R^W_c = \mathbb{1} \qquad \text{(by definition of inverse)}$$

So ...

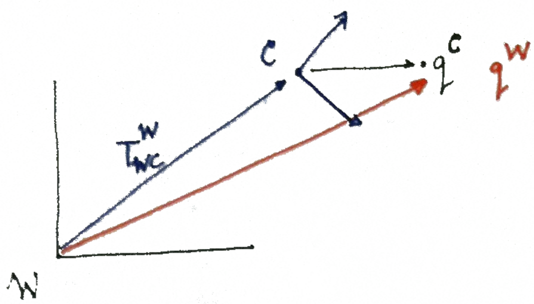
$$(R^W_c)^{-1} = (R^W_c)^T$$

The transpose of a rotation matrix is equal to the inverse of the original rotation matrix.

That means computing the inverse of rotation matrices is easy. Just transpose!

(same holds for the 2D version)

CHANGE OF COORDINATES: ROTATION + TRANSLATION.



What happens when the camera is also translated?

So, there is both a rotation R^W_C and a translation T^W_{WC}

vector translation from W to C given in frame W.

How do I treat the point q^c (in camera frame) to get it written in the world frame?

Well, I need to apply the rotation part then the translation part.

(yes, the order matters)

$$q^W = R^W_C q^c + T^W_{WC} \qquad (*)$$

What about the other way? 1) just flip the letters around

2) use equation (*) to solve for q^c.

version 1) $q^c = R^c_W q^W + T^c_{cW}$

version 2) $$q^W = R^W_c \, q^c + T^W_{wc}$$

$\Rightarrow$

$$q^W - T^W_{wc} = R^W_c \, q^c$$

$\Rightarrow$

$$(R^W_c)^{-1} \left[\cancel{q} \, q^W - T^W_{wc} \right] = q^c$$

$\Rightarrow$

$$q^c = (R^W_c)^T q^W - (R^W_c)^T T^W_{wc}$$

equating the two means:

$$R^c_w = (R^W_c)^T$$ (we knew this)

$$T^c_{cw} = -(R^W_c)^T T^W_{wc}$$ (interesting!)

the coordinate rotation plays a role when trying to invert the direction of a coordinate change transformation!