The Exponential Function for $\mathfrak{so}(3)$

$$R(\omega,\tau) = e^{\hat{\omega}\tau} \qquad \omega \in \mathbb{R}^3 \;/\; \hat{\omega} \in \mathfrak{so}(3)$$

but how do we compute $e^{\hat{\omega}\tau}$?

well, we exploit an interesting property of $\hat{\omega}$.

Lemma. Given $a \in \mathbb{R}^3 \;/\; \hat{a} \in \mathfrak{so}(3)$, then

$$\hat{a}^2 = aa^T - \|a\|^2 \mathbb{1}$$

$$\hat{a}^3 = -\|a\|^2 \hat{a}$$

leads to:

$$\hat{a}^4 = \hat{a}^3\hat{a} = -\|a\|^2 \hat{a}\hat{a} = -\|a\|^2\hat{a}^2$$

$$\hat{a}^5 = \hat{a}^4\hat{a} = -\|a\|^2\hat{a}^3 = \|a\|^4\hat{a}$$

$$\hat{a}^6 = \|a\|^4\hat{a}^2$$

$$\vdots$$

$$\hat{a}^{2k+1} = (-1)^k \|a\|^{2k}\hat{a}$$

$$\hat{a}^{2k+2} = (-1)^k \|a\|^{2k}\hat{a}^2$$

* super important for computing $e^{\hat{\omega}\tau}$ since we are going to use the infinite series expansion for the exp function.

the increasing powers of $\hat{\omega}$ all simplify to $\hat{a}$ or $\hat{a}^2$, and can be factored out!

OK, so if we consider $a = \omega\tau$ in the Lemma,

$$e^{\hat{\omega}\tau} = \sum_{n=0}^{\infty} \frac{1}{n!}(\hat{\omega}\tau)^n = \underbrace{\sum_{k=0}^{\infty} \frac{1}{2k!}(\hat{\omega}\tau)^{2k}}_{\text{even powers}} + \underbrace{\sum_{k=0}^{\infty} \frac{1}{(2k+1)!}(\hat{\omega}\tau)^{2k+1}}_{\text{odd powers}}$$

$$= (\hat{\omega}\tau)^0 + \sum_{k=0}^{\infty} \frac{1}{(2k+2)!}(\hat{\omega}\tau)^{2k+2} + \sum_{k=0}^{\infty} \frac{1}{(2k+1)!}(\hat{\omega}\tau)^{2k+1}$$

$$= \mathbb{1} + \sum_{k=0}^{\infty} \frac{1}{(2k+2)!} \tau^{2k+2} \cdot (-1)^k \cdot \|\omega\|^{2k} \hat{\omega}^2 + \sum_{k=0}^{\infty} \frac{1}{(2k+1)!} \tau^{2k+1} \cdot (-1)^k \|\omega\|^{2k} \hat{\omega}$$

need 2k+2 need 2k+1

$$= \mathbb{1} + \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+2)!} \tau^{2k+2} \|\omega\|^{2k+2} \frac{\hat{\omega}^2}{\|\omega\|^2} + \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} \tau^{2k+1} \|\omega\|^{2k+1} \frac{\hat{\omega}}{\|\omega\|}$$

so add in then divide out

$$= \mathbb{1} + \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{2k!} \|\omega\|^{2k} \tau^{2k} \cdot \frac{\hat{\omega}^2}{\|\omega\|^2} + \left(\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} \|\omega\|^{2k+1} \tau^{2k+1} \right) \frac{\hat{\omega}}{\|\omega\|}$$

this is sine function!

$$= \mathbb{1} + \left(- \sum_{k=1}^{\infty} \frac{(-1)^k}{2k!} \|\omega\|^{2k} \tau^{2k} \right) \frac{\hat{\omega}^2}{\|\omega\|^2} + \frac{\hat{\omega}}{\|\omega\|} \sin(\|\omega\|\tau)$$

this is almost negative cosine. just missing the k=0 term. add in & take out.

$$= \mathbb{1} + \frac{\hat{\omega}^2}{\|\omega\|^2}(1 - \cos(\|\omega\|\tau)) + \frac{\hat{\omega}}{\|\omega\|} \sin(\|\omega\|\tau)$$

$$= \mathbb{1} + \frac{\hat{\omega}}{\|\omega\|} \sin(\|\omega\|\tau) + \frac{\hat{\omega}}{\|\omega\|^2} [1 - \cos(\|\omega\|\tau)]$$ ← Rodrigues' formula.

DONE!

To show how the Lemma was essential, let's look at the SO(2)/SO(2) case, using the complex representation.

Z_R a rotation $\Rightarrow$ $Z_R \bar{Z}_R = \bar{Z}_R Z_R = 1$ unit complex.

$\bar{Z}_R \dot{Z}_R = 0 + j\omega = j\omega$ where ω is the angular velocity.

$(j\omega)^1 = j\omega$
$(j\omega)^2 = -\omega^2$
$(j\omega)^3 = -j\omega^3$
$(j\omega)^4 = \omega^4$
$(j\omega)^5 = j\omega^5$
$(j\omega)^6 = -\omega^6$
$\vdots$
$(j\omega)^{2k+1} = (-1)^k \omega^{2k+1}$
$(j\omega)^{2k+2} = (-1)^{k+1} \omega^{2k}$
$\vdots$

notice the alternating trend similar to SO(3) case.

if $e^{j\omega\tau}$ done as an infinite series it would split into even & odd terms to give

we know that $e^{j\omega\tau} = \cos(\omega\tau) + j\sin(\omega\tau)$

and we get a complex (unit magnitude) number equivalent to a rotation.

a similar property holds for $\hat{\omega} = -J\omega$, the homogeneous matrix form.

$J^0 = \mathbb{1}$
$J^1 = J$
$J^2 = -\mathbb{1}$
$J^3 = -J$
$J^4 = \mathbb{1}$

and the cycle repeats! the $\mathbb{1}$ and J terms lead to cosine and sine, respectively.

Going back to the equation

$$R(\omega, \tau) = e^{\hat{\omega}\tau}$$

angular velocity

quantity of time

rotation matrix describing how frame changes as a function of time.

The Logarithm Function for SO(3)

$$\omega = \ln_\tau R \qquad\qquad R \in SO(3)$$

the time variable τ indicates how long the angular velocity would act for to result in rotation R.

The logarithm has two parts. One for scale, one for axis. Together, they give ω. So, define $\hat{\omega} = \sigma \hat{n}$ OR $\omega = \sigma \vec{n}$

unit vector (axis) hatted

scale (ang. speed)

of rotation

Recall the $e^{\hat{\omega}\tau} = \mathbb{1} + \frac{\hat{\omega}}{\|\omega\|}\sin(\|\omega\|\tau) + \frac{\hat{\omega}^2}{\|\omega\|^2}(1-\cos(\|\omega\|\tau))$

A tricky thing to do is to look at the diagonal & skew-symmetric parts of R.
If $R = e^{\hat{\omega}\tau}$, then

$$\mathrm{Tr}(R) = \mathrm{Tr}\left[\mathbb{1} + \frac{\hat{\omega}}{\|\omega\|}\sin(\|\omega\|\tau) + \frac{\hat{\omega}^2}{\|\omega\|^2}(1-\cos(\|\omega\|\tau))\right]$$

$$= 3 + 0 + \frac{(1-\cos(\|\omega\|\tau))}{\|\omega\|^2}\mathrm{Tr}[\hat{\omega}^2]$$

$$\mathrm{Tr}[\hat{\omega}^2] = \mathrm{Tr}\left[\omega\omega^T - \|\omega\|^2\mathbb{1}\right]$$

$$= \mathrm{Tr}[\omega\omega^T] - \|\omega\|^2\mathrm{Tr}[\mathbb{1}]$$

$$= \|\omega\|^2 - 3\|\omega\|^2$$

$\Rightarrow$

$$= -2\|\omega\|^2$$

$$\mathrm{Tr}(R) = 3 + (1-\cos(\|\omega\|\tau))(-2) = 3 - 2 + 2\cos(\|\omega\|\tau)$$

$$= 1 + 2\cos(\|\omega\|\tau)$$

$\Rightarrow$

$$\sigma = \|\omega\| = \frac{1}{\tau}\cos^{-1}\left(\frac{\mathrm{Tr}(R)-1}{2}\right)$$

Now, to get the axis. Look at the skew-symmetric part.

$$R - R^T = 2 \frac{\hat{\omega}}{\|\omega\|} \sin(\|\omega\|\tau)$$

$\Rightarrow$

$$R - R^T = 2\hat{n} \sin(\sigma\tau)$$ $\leftarrow$ if $\sigma = 0$, then $\vec{n}$ can be anything.

$\Rightarrow$

$$\hat{n} = \frac{R - R^T}{2\sin(\sigma\tau)}$$

$\Rightarrow$ unhat

$$\vec{n} = \frac{(R - R^T)^\vee}{2\sin(\sigma\tau)}$$

$\Rightarrow$

$$\omega = \sigma\vec{n} \qquad \text{if } \sigma \neq 0$$

The exponential takes a constant vector as input.

The logarithm results in a constant vector as output.

For the same time argument, they are inverses of each other.

$$\ln_\tau (e^{\hat{\omega}\tau}) = \hat{\omega}$$

$$e^{(\ln_\tau R)\tau} = R$$