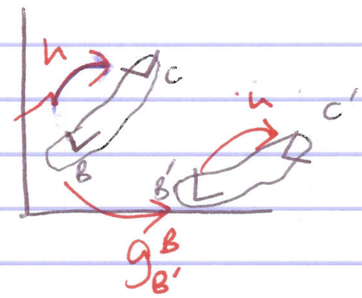
ECE 4560 09/10/07

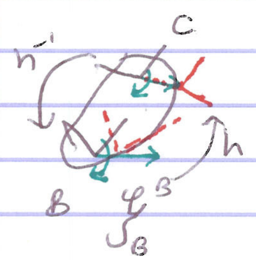


With respect to frame B to configuration B'. It does so under some constant body velocity

$$\xi^B_B = \begin{Bmatrix} v \\ \omega \end{Bmatrix} \qquad v = \text{constant}, \quad \omega = \text{constant}$$

ok, but what does the reference frame C experience?

$$\xi^C_C = h^{-1} \xi^B_B h$$



$$h = h^B_C$$

$$\Rightarrow \xi^C_C = h^C_B \, \xi^B_B \, h^B_C$$

In homogeneous coordinates:

$$\hat{\xi}^C_C = h^{-1} \hat{\xi}^B_B h = \begin{bmatrix} R_h & d_h \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} \hat{\omega} & v \\ 0 & 0 \end{bmatrix} \begin{bmatrix} R_h & d_h \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} R_h^T & -R_h^T d_h \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{\omega} & v \\ 0 & 0 \end{bmatrix} \begin{bmatrix} R_h & d_h \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} R_h^T \hat{\omega} & R_h^T v \\ 0 & 0 \end{bmatrix} \begin{bmatrix} R_h & d_h \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} R_h^T \hat{\omega} R_h & R_h^T \hat{\omega} d_h + R_h^T v \\ 0 & 0 \end{bmatrix}$$

$$R_h^T \hat{\omega} R_h = -\omega \underbrace{R_h^T J R_h}$$

$$= -\omega J = \hat{\omega}$$

$$\hat{\xi}_c^c = \left[\begin{array}{c|c} \hat{\omega} & R_h^T \hat{\omega} d_h + R_h^T v \\ \hline 0 & 0 \end{array}\right]$$

$\Rightarrow$ So, homogeneous coordinates, we can just ~~multiply~~ do matrix multiplication, what if I wanted a formula for the adjoint in vector form?

$$\xi_c^c = \begin{Bmatrix} v_c^c \\ \omega_c^c \end{Bmatrix} = Ad_{h^{-1}} \begin{Bmatrix} v_B^B \\ \omega_B^B \end{Bmatrix}$$

First I want to know what $Ad_h \xi$ is, then worry about the inverse.

$$Ad_h \hat{\xi} = h \hat{\xi} h^{-1} \qquad \text{where } \xi = \begin{Bmatrix} v \\ \omega \end{Bmatrix}$$

$$= \begin{bmatrix} R_h & d_h \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{\omega} & v \\ 0 & 0 \end{bmatrix} \begin{bmatrix} R_h^T & -R_h^T d_h \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} R_h \hat{\omega} R_h^T & -R \hat{\omega} R^T d_h + R v \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \hat{\omega} & -\hat{\omega} d_h + R v \\ 0 & 0 \end{bmatrix}$$

My interest is <u>vector form</u> so I unhat

$$(Ad_h \hat{\xi})^\vee = \begin{Bmatrix} -\hat{\omega} d_h + R_h v \\ \omega \end{Bmatrix}$$

Can I get $(Ad_h \hat{\xi}) =$ MATRIX $\cdot \, \xi$

$\longrightarrow$

$$= \begin{Bmatrix} \omega J^{\top} d_h + R_h v \\ \omega \end{Bmatrix} = \left[\begin{array}{c|c} R & J^{\top} d_h \\ \hline 0 & 1 \end{array}\right] \begin{Bmatrix} v \\ \omega \end{Bmatrix}$$

So, in vector form for ξ,

$$\text{☆} \quad Ad_h \xi = \begin{bmatrix} R_h & J^{\top} d_h \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} v \\ \omega \end{Bmatrix} \quad \text{for } \xi = \begin{Bmatrix} v \\ \omega \end{Bmatrix}$$

↳ Back to our story

$$Ad_{h^{-1}} \xi_B^B = \begin{bmatrix} R_h^T & -J^{\top} R_h^T d_h \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} v \\ \omega \end{Bmatrix} = \begin{Bmatrix} R_h^T v - J^{\top} R_h^T \omega d_h \\ \omega \end{Bmatrix}$$

$$h^{-1} = \left[\begin{array}{c|c} R_h^T & -R_h^T d_h \\ \hline 0 & 1 \end{array}\right]$$

Almost agrees, except for ordering of $J^{\top}$ and R_h^T but,

$$R_h J^{\top} R_h^T = J^{\top}$$

$\Rightarrow$ mult. by R_h^T on the left

$$\cancel{R_h^T R_h} J^{\top} R_h^T = R_h^T J^{\top}$$

$$J^{\top} R_h^T = R_h^T J^{\top}$$

so, it actually agrees.

<u>SYNOPSIS</u>:

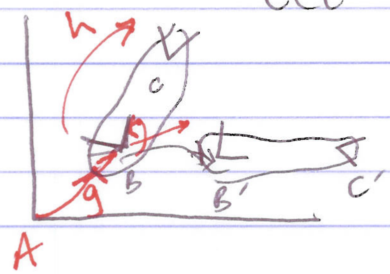
$$Ad_h \xi = \begin{bmatrix} R_h & J^{\top} d_h \\ 0 & 1 \end{bmatrix} \xi \quad \text{in vector form} \quad \xi = \begin{Bmatrix} v \\ \omega \end{Bmatrix}$$

OR

$$Ad_h \hat{\xi} = h \hat{\xi} h^{-1} \quad \text{in homogeneous form.}$$

ECE 4560

09/12/07



Body velocity : ξ_B^B

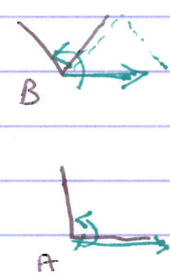
↑ velocity of body w.r.t coordinate frame of the body.

Suppose, I would rather know the velocity of another point on the body whose relative configuration is h (from B to C)

$$\xi_C^C = Ad_{h^{-1}} \xi_B^B \quad \text{where } h = h_C^B$$

Adjoint didnot change rotational velocity.
For SE(2) (e.g. planar case), rotations do not affect angular velocity.

$$\xi_B^A = g_B^A * \xi_B^B$$



$$\xi_B^B = \begin{Bmatrix} 1 \\ -1 \\ 1 \end{Bmatrix} \qquad \xi_B^A = \begin{Bmatrix} \sqrt{2} \\ 0 \\ 1 \end{Bmatrix}$$

$$= \left[\begin{array}{c|c} R & \mathbb{J}d \\ \hline 0 & 1 \end{array}\right] \begin{Bmatrix} v \\ w \end{Bmatrix}$$

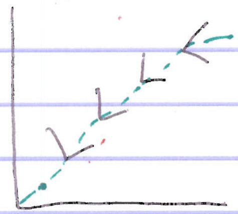
$$\xi_B^A = \begin{Bmatrix} Rv \\ w \end{Bmatrix}_B^A$$

How to change reference frames of a vector's coordinate representation.

$$\boxed{Ad_h = \begin{bmatrix} R_h & \mathbb{J} \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} v \\ w \end{Bmatrix}}$$

$$R\vec{v} + \mathbb{J}w$$
$$= R\begin{Bmatrix} v_x \\ v_y \end{Bmatrix} + \begin{Bmatrix} d_x \\ d_y \end{Bmatrix} w$$

Why do I need this?
Because we may be given ξ_B^A and we really want ξ_B^B



$g(t) = g_B^A(t)$

$\Rightarrow \frac{d}{dt}$

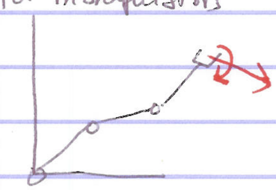
$\dot{g}_B^A(t)$ ← this is $\xi_B^A(t)$

How did I get into body frame?

We did $\xi_B^B(t) = g_A^B(t) * \xi_B^A(t)$ (on page 8-1)

$= \left(g_B^A(t)\right)^{-1} * \xi_B^A(t)$

for manipulators



$g_B^A = g_B^A(\theta_1, \dots \theta_n)$

Spatial Velocity:

defined as: $\dot{g}(t)\, g^{-1}(t)$ — $\left\{ \dot{g}_B^A(t) \left(g_B^A(t)\right)^{-1} \right.$

$= \dot{g}_B^A(t)\, g_A^B(t)$

$= \xi_A^A(t)$

what is this? wtf?!!

What if I do the following:

$Ad_{(g_B^A)^{-1}} \xi_B^B = (g_B^A)\, \xi_B^B\, (g_B^A)^{-1} =$

$\underbrace{g_A^B\, \xi_B^A}_{}$ $\dot{g}_B^A$

$= \cancel{g_B^A\, g_A^B}\, \dot{g}_B^A\, g_A^B$

The spatial velocity is the velocity associated ~~with~~ to the observer frame (A) moving as though it were rigidly attached to the body frame (B), which itself was moving.

$$\xi_A^A = \mathrm{Ad}_{g^{-1}}\, \xi_B^B \qquad \text{where } g = g_B^A$$

So, what is it for SE(2)?

$$\dot{g}_B^A(t)\left(g_B^A(t)\right)^{-1} = \left[\begin{array}{c|c} \Delta R(\theta(t))\,\dot{\theta}(t) & \dot{d}(t) \\ \hline 0 & 0 \end{array}\right]\left[\begin{array}{c|c} R^T(\theta(t)) & -R^T(\theta(t))\,d(t) \\ \hline 0 & 1 \end{array}\right]$$

$$\text{for } g_B^A(t) = \left[\begin{array}{c|c} R(\theta(t)) & d(t) \\ \hline 0 & 1 \end{array}\right]$$

$$= \left[\begin{array}{c|c} DR(\theta(t))R^T(\theta(t))\,\dot{\theta} & -DR(\theta(t))\cdot R^T(\theta(t))\,d(t)\,\dot{\theta}(t) + \dot{d}(t) \\ 0 & 0 \end{array}\right]$$

$$= \left[\begin{array}{c|c} -J^T\dot{\theta} & J^T d\,\dot{\theta} + \dot{d} \\ \hline 0 & 0 \end{array}\right]$$

$$\hat{\xi}_A^A(t) = \left[\begin{array}{c|c} \hat{\dot{\theta}} & -\hat{\dot{\theta}}\,d + \dot{d} \\ \hline 0 & 0 \end{array}\right]$$

Homogeneous form of spatial velocity.
velocity of frame A moving in a manner consistent with being rigidly attached to the moving frame B.

ECE 4560 09/14/07

Vectors of SE(2). Discussed:

1) Linear Structure
2) Adjoint & change of reference frame for a moving body

TO DO: 3) Relationship b/w elements and vectors (velocities)
Mapping of displacement to vector and vice-versa

3) Exponential Representation

Notation:

a configuration is given by a Lie group element.
$g \in SE(2)$
↳ special Euclidean

a vector associated to SE(2) is given by a Lie algebra element, $\xi \in \mathfrak{se}(2)$

We want the equivalent of

$$\dot{x} = v$$

for g and ξ

In SE(2), ξ looks like $\xi = \begin{Bmatrix} v \\ \omega \end{Bmatrix}$

Given $\dot{g} = g * \xi$

what happens if $\omega = 0$

$$\Rightarrow \dot{g} = \begin{bmatrix} R & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} v \\ \omega \end{Bmatrix} = \begin{Bmatrix} Rv \\ 0 \end{Bmatrix}$$

$$\Rightarrow \begin{aligned} \dot{x} &= R_{11} v_x + R_{12} v_y = w_x \\ \dot{y} &= R_{21} v_x + R_{22} v_y = w_y \\ \dot{\theta} &= 0 \end{aligned}$$

$R = \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix}$

$$\Rightarrow \quad \dot{x} = W_x$$
$$\dot{y} = W_y$$
$$\dot{\theta} = 0$$

What happens when $\omega \neq 0$?
In this case,

$$\dot{x} = R_{11}(\theta) v_x + R_{12}(\theta) v_y$$
$$\dot{y} = R_{21}(\theta) v_x + R_{22}(\theta) v_y$$
$$\dot{\theta} = \omega$$

$$\Rightarrow \quad \dot{x} = v_x \cos\theta \mp v_y \sin\theta$$
$$\dot{y} = + v_x \sin\theta + v_y \cos\theta$$
$$\dot{\theta} = \omega$$

Is there a solution to this?

Justify my solution:
Recall : $\dot{g} = g \cdot \xi$

In linear control, you guys worked with:

$$\dot{x} = Ax$$

what was the integral?
Solution was an exponential,

$$x(t) = e^{At} x_0$$

Basic definition: $e^{At} = \sum_{n=1}^{\infty} \frac{(At)^n}{n!}$

Best approach is to diagonalize:

$$A = T \Sigma T^{-1} = Ad_T \Sigma$$

$$e^A = e^{Ad_T \Sigma} = Ad_T e^{\Sigma} = T e^{\Sigma} T^{-1}$$

$$e^{\begin{bmatrix} \sigma_1 & & 0 \\ & \ddots & \\ 0 & & \sigma_n \end{bmatrix}} = \begin{bmatrix} \sigma_1 & & 0 \\ & \ddots & \\ 0 & & \sigma_n \end{bmatrix}$$

In the end I get a transition matrix Φ, $x_{new} = \Phi \, x_{old}$

For elements of $\mathfrak{se}(2)$,
there is an exponential also.

It is defined:

$$\exp(\hat{\xi}\tau) = \begin{cases} \text{if } \omega = 0 & \text{then it is } \begin{bmatrix} \mathbb{1} & v\tau \\ 0 & 1 \end{bmatrix} \\ \text{if } \omega \neq 0 & \text{then it is } \begin{bmatrix} e^{\hat{\omega}\tau} & -\frac{1}{\omega}(\mathbb{1} - e^{\hat{\omega}\tau}) J v \\ 0 & 1 \end{bmatrix} \end{cases}$$

$\exp : \mathfrak{se}(2) \times \mathbb{R} \to SE(2)$

The inverse operation is called the logarithm, and will be denoted by $\ln$.

It has one of two forms:

$\ln : SE(2) \to \mathfrak{se}(2)$ (time not specified therefore implied to be 1)

OR

$\ln : SE(2) \to \mathfrak{se}(2) \times \mathbb{R}$ (time specified)

Given $g \in SE(2)$, then

$g = \begin{bmatrix} R & d \\ 0 & 1 \end{bmatrix}$

$$(\xi, \tau) = \ln g = \begin{cases} \text{if } R = \mathbb{1}, \text{ then } v = \frac{d}{\|d\|} \ \& \ \tau = \|d\|, \ \omega = 0 \\ \text{else,} \quad \tau = \operatorname{atan2}(R_{21}, R_{11}) \\ \qquad \omega = 1 \\ \qquad v = J^{T}(\mathbb{1} - R)^{-1} d \end{cases}$$

The alternative is

$$\xi = \ln g = \begin{cases} \text{if } R = \mathbb{1}, \text{ then} & v = d \\ & \omega = 0 \\ \text{else} & \omega = \text{atan2}(R_{21}, R_{11}) \\ & v = \omega J^{-1} (\mathbb{1} - R)^{-1} d \end{cases}$$