ECE 4560 09/24/2007

MANIPULATORS & MANIPULATOR ANALYSIS:

- Joints
- Workspace, Configuration Space, Joint Space

$$Q = M \times G$$

- Workspace Analysis

Actuator Space Joint Space (shape space) Configuration space (Group space)

inverse kinematics

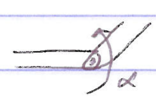
forward kinematics

Forward Kinematics

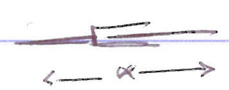
- Configuration of end-effector given joint configuration of manipulator
- gives transformation from manipulator base frame (typically) to end-effector frame.
- done by concatening transformations that go from joint to joint.

Joints are traditionally chosen from a set of 6 simpler ones, called lower-pair joints.

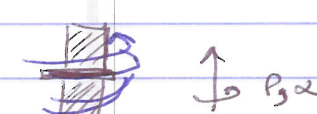
1) revolute



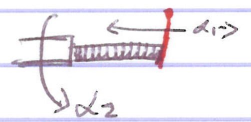
2) prismatic



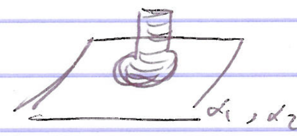
3) helical/screw



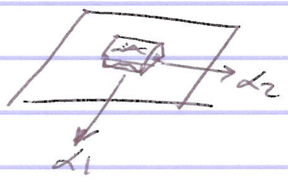
4) cylindrical



5) Spherical / ball & joint



6) planar



1, 2, 3 most common
3, 6 least common.

- Simple nature of lower-pair joints allows for product of Lie groups / product of exponentials formula.

- If we consider each joint to have 1 degree of freedom, then the joint space is products of
  1) S^1 (revolute joint) S^1 - circle
  , or
  2) $\mathbb{R}$ (prismatic or helical joint)

Products of S^1 gives a torus, $T^r = \underbrace{S^1 \times \dots \times S^1}_{r \text{ copies}}$

$\mathbb{R}$ gives $\mathbb{R}^p$ $\quad \mathbb{R}^p = \underbrace{\mathbb{R} \times \dots \times \mathbb{R}}_{p\text{-copies}}$

$\Rightarrow$ joint space is $M = T^r \times \mathbb{R}^p$

$r = \#$ of revolute joints
$p = \#$ of prismatic joints

Workspace description:

Workspace (complete)

$$W = \{ g_e(\alpha) \in SE(n) \mid \alpha \in M \}$$

- set of all end-effector configurations reachable by some joint configuration.

Reachable Workspace:

$$W_R = \{ p_e(\alpha) \in E(n) \mid \alpha \in M \}$$

$E(n) \equiv \mathbb{E}^n = \mathbb{R}^n$

- set of positions reachable with arbitrary orientation ($g_e(\alpha) = (R_e(\alpha), p_e(\alpha))$).

* not always useful since orientation not always controllable.

Dextrous Workspace:

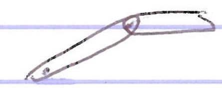
$$W_D = \{ p_e [redacted] \in E(n) \mid \forall R \in SO(n), \exists \alpha \in M : g_e = (R, p_e) \}$$

- set of all positions achievable with arbitrary orientation.

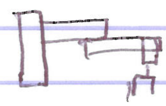
(added 09/26) * within this volume, we can do anything (full control)
* typically, to maximize the dextrous workspace, industrial manipulators add a "spherical" wrist to the end of the manipulator chain.

Example of maximizing dextrous workspace:

SCARA manipulator adds a cylindrical joint for full SE(2) control.



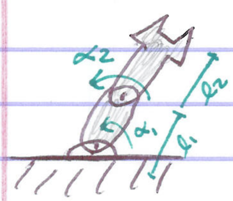
SCARA (Top view)



side view.

Example 1 (kinematically insufficient manipulator)

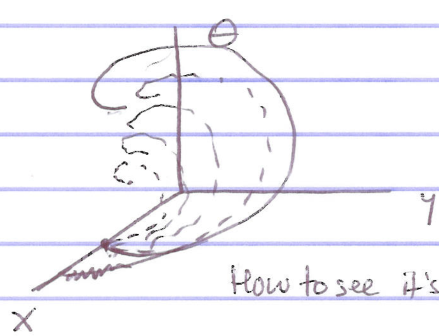
$$g_e(\alpha) = \begin{Bmatrix} l_1 \cos(\alpha_1) + l_2 \cos(\alpha_1 + \alpha_2) \\ l_1 \sin(\alpha_1) + l_2 \sin(\alpha_1 + \alpha_2) \\ \alpha_1 + \alpha_2 \end{Bmatrix}$$



Note: $\dim(M) < \dim(G)$

M: joint space, G: configuration space.

Full workspace W:
- will be a surface in SE(3)



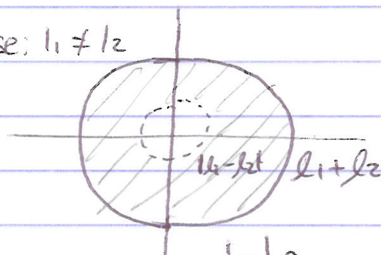
- has area but no volume (system is not fully controllable)
- kinda rare to see since it is tough to visualize

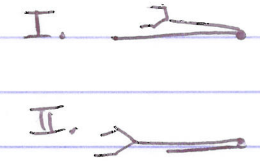
How to see it's a surface?

Reachable Workspace W_R:

- depends on the geometry of the manipulator
  1. actuator limits on α_1 & α_2.
  2. link lengths.

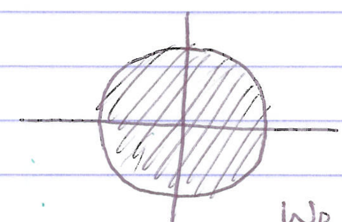
Case: $l_1 \neq l_2$





W_R = the annulus $|l_1 - l_2| \leq r \leq l_1 + l_2$

Case: $l_1 = l_2 = l$



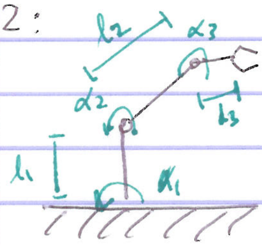
W_R = disc of radius $r = l_1 + l_2$.

Dextrous Workspace W_D:

if $l_1 \neq l_2$
$W_D = \emptyset$ (empty set)

if $l_1 = l_2$
$W_D = \{(0,0)\}$.

Example 2:



assume $l_1 > l_2 > l_2$

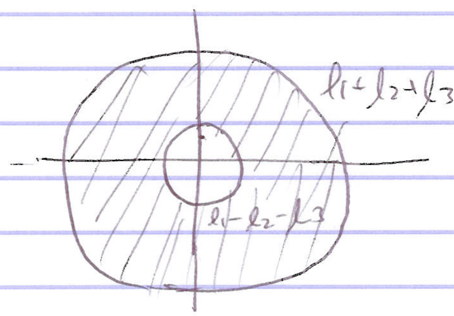
$l_1 > l_2 + l_3$

$$g_e(\alpha) = \begin{Bmatrix} l_1 \cos(\alpha_1) + l_2 \cos(\alpha_1+\alpha_2) + l_3 \cos(\alpha_1+\alpha_2+\alpha_3) \\ l_1 \sin(\alpha_1) + l_2 \sin(\alpha_1+\alpha_2) + l_3 \sin(\alpha_1+\alpha+\alpha_3) \\ \alpha_1+\alpha_2+\alpha_3 \end{Bmatrix}$$

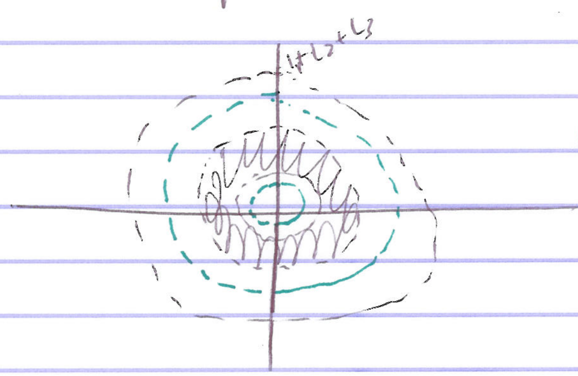
$\dim(M) = \dim(G)$

$\Rightarrow$ should have a non-trivial workspace.

Reachable Workspace W_R = annulus with $l_1 - l_2 - l_3 \leq r \leq l_1 + l_2 + l_3$



Dextrous Workspace, W_D = annulus $l_1 - l_2 + l_3 \leq r \leq l_1 + l_2 - l_3$



Compared to W_R, W_D is "smaller" by $2l_3$ due to larger inner radius (by l_3) and smaller outer radius. (by l_3)

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Forward kinematics:

- Product of Lie groups (body)
- Product of Exponentials (spatial)

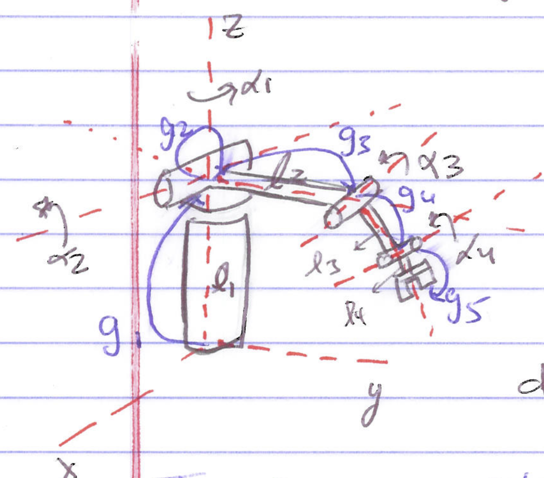
Definition. The forward kinematics of a manipulator is the configuration of the end-effector given a joint configuration of the manipulator.

Product of Lie groups: $g_e(\alpha) = g_1(\alpha) \cdots g_n(\alpha_n)\, g_{n+1}$

Product of Exponentials: $g_e(\alpha) = \underbrace{e^{\xi_1 \alpha_1} \cdots e^{\xi_n \alpha_n}}_{\text{act like single degree of freedom lower-pair joints.}} g_0$

these are both constants and different from one another.

Example (Product of Lie groups)



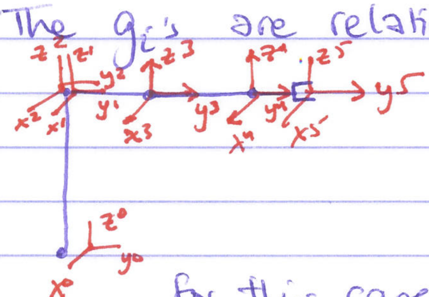
$g_e(\alpha) = g_1(\alpha_1)\, g_2(\alpha_2)\, g_3(\alpha_3)\, g_4(\alpha_4)\, g_5$

GOAL: find $d_i(\alpha_i)$ & $R_i(\alpha_i)$ for $i = 1,2,3,4$

$d_1 = \begin{Bmatrix} 0 \\ 0 \\ l_1 \end{Bmatrix}$ $d_2 = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$ $d_3 = \begin{Bmatrix} 0 \\ l_2 \\ 0 \end{Bmatrix}$

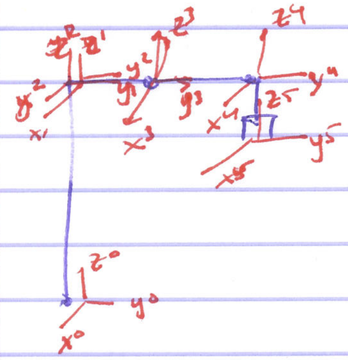
continued

The g_i's are relative to the home position



$d_4 = \begin{Bmatrix} 0 \\ l_3 \\ 0 \end{Bmatrix}$ $d_5 = \begin{Bmatrix} 0 \\ l_4 \\ 0 \end{Bmatrix}$

for this case: $d_5 = \begin{Bmatrix} 0 \\ l_4 \\ 0 \end{Bmatrix}$



for this case:

$$d_5 = \begin{Bmatrix} 0 \\ 0 \\ -l_4 \end{Bmatrix}$$

$$R_1(\alpha_1) = \begin{bmatrix} \cos(\alpha_1) & -\sin(\alpha_1) & 0 \\ \sin(\alpha_1) & \cos(\alpha_1) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$ rotation about z^0-axis

$$R_2(\alpha_2) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\alpha_2) & -\sin(\alpha_2) \\ 0 & \sin(\alpha_2) & \cos(\alpha_2) \end{bmatrix}$$ rotation about x^1-axis.

$$R_3(\alpha_3) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\alpha_3) & -\sin(\alpha_3) \\ 0 & \sin(\alpha_3) & \cos(\alpha_3) \end{bmatrix}$$ rotation about x^2-axis.

$$R_4(\alpha_4) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\alpha_4) & -\sin(\alpha_4) \\ 0 & \sin(\alpha_4) & \cos(\alpha_4) \end{bmatrix}$$ rotation about x^3-axis

$$R_5 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Now, we'll slowly build up to $g_e = g_1 g_2 g_3 g_4 g_5$

now, $g_2(\alpha_2)g_3(\alpha_3)g_4(\alpha_4) = \left[\begin{array}{c|c} R_2(\alpha_2)R_3(\alpha_3)R_4(\alpha_4) & \underbrace{d_2 + R_2(\alpha_2)d_3 + R_2(\alpha_2)R_3(\alpha_3)d_4}_{\text{rewritten}} \\ \hline 0 & 1 \end{array}\right]$

$d_2 + R_2(\alpha_2)d_3 + R_2(\alpha_2)R_3(\alpha_3)d_4$

$$= \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & & \cos(\alpha_2+\alpha_3+\alpha_4) & -\sin(\alpha_2+\alpha_3+\alpha_4) & l_1\cos(\alpha_2) + l_2\cos(\alpha_2+\alpha_3) \\ 0 & & \sin(\alpha_2+\alpha_3+\alpha_4) & \cos(\alpha_2+\alpha_3+\alpha_4) & l_1\sin(\alpha_2) + l_2\sin(\alpha_2+\alpha_3) \\ \hline 0 & 0 & 0 & & 1 \end{array}\right]$$

Next:

$$g_2(\alpha_2)g_3(\alpha_3)g_4(\alpha_4)g_5 = \left[\begin{array}{ccc|c} 1 & 0 & 0 & \\ 0 & \cos(\alpha_2+\alpha_3+\alpha_4) & -\sin(\alpha_2+\alpha_3+\alpha_4) & \\ 0 & \sin(\alpha_2+\alpha_3+\alpha_4) & \cos(\alpha_2+\alpha_3+\alpha_4) & \\ \hline 0 & 0 & 0 & 1 \end{array}\right]$$

only a translation ~~only a rotation~~
therefore ~~the~~
rotation component doesn't change

$$\begin{array}{c} 0 \\ l_1\cos(\alpha_2) + l_2\cos(\alpha_2+\alpha_3) + l_3(\alpha_2+\alpha_3+\alpha_4) \\ l_1\sin(\alpha_2) + l_2\sin(\alpha_2+\alpha_3) + l_3(\alpha_2+\alpha_3+\alpha_4) \end{array}$$

$\Rightarrow$ Next: $g_e(\alpha) = g_1(\alpha_1)g_2(\alpha_2)g_3(\alpha_3)g_4(\alpha_4)g_5$