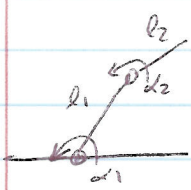
The Geometric Approach:

- by observation, reduce geometry of the linkage to easily solved sub-problems
  - one technique reduces the linkage to triangle.

Example:

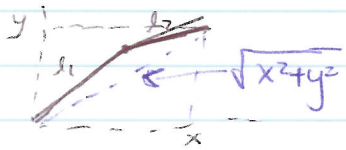
Many manipulators have the following sub-problem, which is usually obtained by projecting the manipulator geometry to 2 coordinate axes.



$$\begin{Bmatrix} x \\ y \\ \theta \end{Bmatrix} = \begin{Bmatrix} l_1 \cos\alpha_1 + l_2 \cos(\alpha_1+\alpha_2) \\ l_1 \sin\alpha_1 + l_2 \sin(\alpha_1+\alpha_2) \\ \alpha_1+\alpha_2 \end{Bmatrix}$$

for this sub-problem only have 2 degrees of freedom
$\Rightarrow$ cannot place with arbitrary position & orientation.

$\Rightarrow$ so, just worry about positional placement.

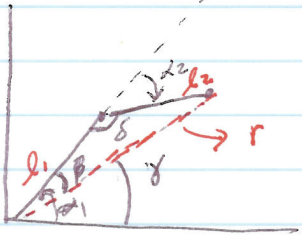


To have solution, check:

$$l_1 + l_2 \geq \sqrt{x_{des}^2 + y_{des}^2} \geq |l_1 - l_2|$$

(desired position should lie within annulus defined by reachable workspace for this subproblem)

If all is OK, then use law of cosines

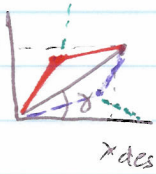


$$\alpha_1 = \beta + \gamma$$
$$\alpha_2 = \pi - \delta$$

$\Rightarrow$

Law of cosines: $r^2 = l_1^2 + l_2^2 - 2 l_1 l_2 \cos(\delta)$

$$\Rightarrow \quad \cos(\delta) = \frac{l_1^2 + l_2^2 - r^2}{2 l_1 l_2}$$

for γ, we have: 

$$\Rightarrow \gamma = \text{atan}(x_{des}, y_{des})$$

note: atan is NOT atan2. for atan2, use $\gamma = \text{atan2}(y_{des}, x_{des})$

for β, use law of cosines with different ordering:

$$l_2^2 = l_1^2 + r^2 - 2 l_1 r \cos(\beta)$$

$$\Rightarrow \quad \cos(\beta) = \frac{l_1^2 + r^2 - l_2^2}{2 l_1 r}$$

There is a little problem. Inverse of cosine is not unique (as shown in above figure. there are 2 shown ways to get to x_{des} and y_{des}) (red and blue)

There are four possible solutions, two of which actually work.

So, solution obtained from:

$\alpha_1 = \gamma + \beta$, $\gamma - \beta$

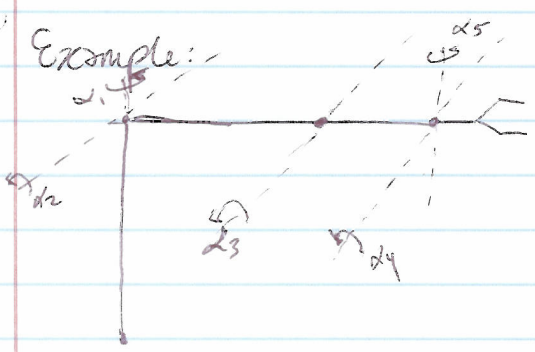
$\alpha_2 = \pi - \delta$, $\pi + \delta$

TO Check: verify orientation

$\alpha_1 + \alpha_2 = \gamma + \beta + \pi - \delta$

$\gamma - \beta + \pi + \delta$

just pick one of the two valid ones.

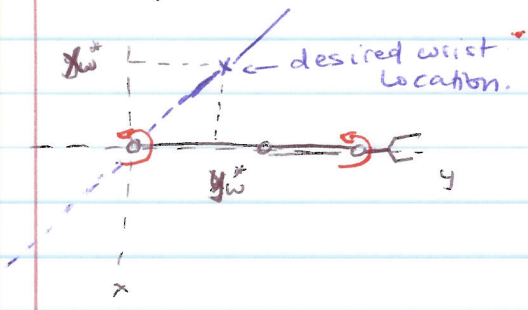
Example: 

How to apply to more complex example.

1) break the problem down to
   i) position of wrist
   ii) orientation of hand/wrist
   iii) angle of first joint.

assume (ii) solved enough to give us desired position of wrist.

① Top view



Solution is $\alpha_1 = \operatorname{atan}(x_w^*, y_w^*) - \frac{\pi}{2}$

since reference configuration is aligned with y-axis & not x-axis.

to be continued....