ECE 4560

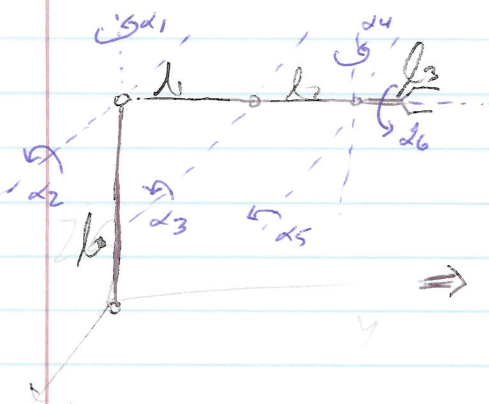
10/15/2007
23-1

last part of inverse kinematics:

get final angle such that $\theta_{des} = \alpha_1 + \alpha_2 + \alpha_3$

$\Rightarrow \alpha_3 = \theta_{des} - \alpha_1 - \alpha_2$

Inverse kinematics of Elbow manipulator (Algebraic Method)



- this approach uses Piepe's method.

$$g_e(\alpha) = g_1(\alpha_1) g_2(\alpha_2) g_3(\alpha_3) g_4(\alpha_4) g_5(\alpha_5) g_6(\alpha_6) g_7$$

$\Rightarrow$ split

$$g_e(\alpha) = \underbrace{g_1(\alpha_1) g_2(\alpha_2) g_3(\alpha_3) \bar{g}_4}_{\text{can position anywhere in the appropriate workspace; no control over orientation}} \underbrace{\tilde{g}_4(\alpha_4) g_5(\alpha_5) g_6(\alpha_6)}_{\text{allows for arbitrary reorientation (no translation)}} \underbrace{g_7}_{\text{constant transfer motion.}}$$

We have some desired end-effector configuration g_e^*.
Move constant g_7 term to right hand side

$$\underbrace{g_e^* g_7^{-1}}_{\text{define } g_w^* = g_e^* g_7^{-1}} = g_1(\alpha_1) g_2(\alpha_2) g_3(\alpha_3) \bar{g}_4 \tilde{g}_4(\alpha_4) \tilde{g}_5(\alpha_5) \tilde{g}_6(\alpha_6)$$

find $(\alpha_1, \alpha_2, \alpha_3)$ such that

$$\text{position}(g_w^*) = \text{position}(g_1(\alpha_1) g_2(\alpha_2) g_3(\alpha_3) \bar{g}_4)$$

If we define

$$g_w(\alpha) = g_1(\alpha_1)\, g_2(\alpha_2)\, g_3(\alpha_3)\, g_4$$

then

$$g_w(\alpha) = \left[\begin{array}{c|c} R_w(\alpha_1,\alpha_2,\alpha_3) & d_w(\alpha_1,\alpha_2,\alpha_3) \\ \hline 0 & 1 \end{array}\right]$$

and

$$\text{position}(g_w(\alpha)) = d_w(\alpha_1,\alpha_2,\alpha_3) = \begin{Bmatrix} x_w \\ y_w \\ z_w \end{Bmatrix}$$

$$\begin{Bmatrix} x_w \\ y_w \\ z_w \end{Bmatrix} = \begin{Bmatrix} -(l_1\cos(\alpha_2) + l_2\cos(\alpha_2+\alpha_3))\sin(\alpha_1) \\ (l_1\cos(\alpha_2) + l_2\cos(\alpha_2+\alpha_3))\cos(\alpha_1) \\ l_0 + l_1\sin(\alpha_2) + l_2\sin(\alpha_2+\alpha_3) \end{Bmatrix}$$

can shift origin up to remove.

Algebraic Approach:

- try to solve simultaneous equations
- elimination theory.
  - successively eliminate variables until a single equation in one unknown is achieved.
  - for manipulator forward kinematics equations, there exists a systematic approach to carry out this procedure.
  - back-substitution gives solution.

$$(x_w^*)^2 + (y_w^*)^2 + (z_w^* - l_0)^2 = x_w^2(\alpha_1,\alpha_2,\alpha_3) + y_w^2(\alpha_1,\alpha_2,\alpha_3) + (z_w(\alpha_1,\alpha_2,\alpha_3) - l_0)^2$$

$$= l_1^2 + l_2^2 - 2 l_1 l_2 \cos(\alpha_3)$$

$$\Rightarrow \cos(\alpha_3) = \frac{(x_w^*)^2 + (y_w^*)^2 + (z_w^* - l_0)^2 - l_1^2 - l_2^2}{2 l_1 l_2}$$

→ two solutions to inverse cosine.

next, we also should have $\alpha_1, \alpha_2, \alpha_3$ satisfy:

I want to find $(\alpha_1, \alpha_2, \alpha_3)$ such that:

$$(x_w^*)^2 + (y_w^*)^2 = x_w^2(\alpha_1, \alpha_2, \alpha_3) + y_w^2(\alpha_1, \alpha_2, \alpha_3)$$

$$= (l_1 \cos(\alpha_2) + l_2 \cos(\alpha_2 + \alpha_3))^2$$

define $r = \sqrt{(x_w^*)^2 + (y_w^*)^2}$

$\Rightarrow$ $r = (l_1 + l_2 \cos(\alpha_3)) \cos(\alpha_2) - (l_2 \sin(\alpha_3)) \sin(\alpha_2)$

what is α_2? we know α_3 already.

apply change of variable using $w = \tan(\frac{1}{2}\alpha_2)$

[because $\cos(\alpha_2) = \frac{1 - w^2}{1 + w^2}$ and $\sin(\alpha_2) = \frac{2w}{1 + w^2}$]

$$\Rightarrow r = (l_1 + l_2 \cos(\alpha_3)) \frac{1 - w^2}{1 + w^2} - \frac{(2 l_2 \sin(\alpha_3)) w}{1 + w^2}$$

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continued from Monday

$$(1+w^2)r = (l_1 + l_2\cos(\alpha_3))(1-w^2) - 2l_2\sin(\alpha_3)w$$

$$\Rightarrow (r+l_1+l_2\cos(\alpha_3))w^2 \cancel{+} 2l_2\sin(\alpha_3)w + (r-l_1-l_2\cos(\alpha_3)) = 0$$

quadratic equation, can be solved using quadratic formula.

given solution for α_3, can solve for w and then use that to solve for α_2 using $\tan(\frac{1}{2}\alpha_2) = w$

→ solve using both solutions to α_3.

→ the quadratic formula gives two solutions per equation there are four potential solutions.

let's worry about these later.

Lastly, to get α_1, we want to find the solution to

$$\sin(\alpha_1) = -\frac{x_w^*}{l_1\cos(\alpha_2) + l_2\cos(\alpha_2+\alpha_3)}$$

$$\cos(\alpha_1) = \frac{y_w^*}{l_1\cos(\alpha_2) + l_2\cos(\alpha_2+\alpha_3)}$$

$$\alpha_1 = \mathrm{atan2}(\cos(\alpha_1), \sin(\alpha_1)) = \mathrm{atan2}(y_w^*, x_w^*)$$

One solution

Now, go back to the four solutions & pick a correct one.

do that by plugging it in and verifying that

$$x_w^* = x_w(\alpha_1, \alpha_2, \alpha_3)$$
$$y_w^* = y_w(\alpha_1, \alpha_2, \alpha_3)$$
$$z_w^* = z_w(\alpha_1, \alpha_2, \alpha_3)$$

* once the wrist is positioned, the next step is to get orientation correct.

$$g_e^* = \underbrace{g_1(\alpha_1^*) g_2(\alpha_2^*) g_3(\alpha_3^*) g_4}\ \tilde{g}_4(\alpha_4) g_5(\alpha_5) g_6(\alpha_6) g_7$$

$$g_w^{-1}(g_e^* = g_w \tilde{g}_4(\alpha_4) g_5(\alpha_5) g_6(\alpha_6) g_7)\, g_7^{-1}$$

$$g_w^{-1} g_e^* g_7^{-1} = \tilde{g}_4(\alpha_4) g_5(\alpha_5) g_6(\alpha_6)$$

$$g_w^{-1} g_w^* = \tilde{g}_4(\alpha_4) g_5(\alpha_5) g_6(\alpha_6)$$

If our work is correct, then

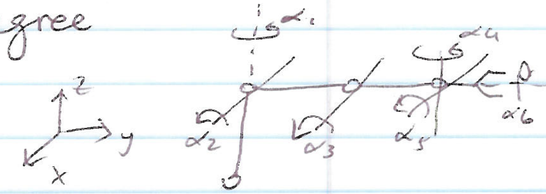
$$g_w^{-1}(\alpha_1, \alpha_2, \alpha_3)\, g_w^* = \begin{bmatrix} R & 0 \\ 0 & 1 \end{bmatrix}$$

and we know that

$$\tilde{g}_4(\alpha_4) g_5(\alpha_5) g_6(\alpha_6) = \begin{bmatrix} R(\alpha_4, \alpha_5, \alpha_6) & 0 \\ 0 & 1 \end{bmatrix}$$

so, we just need to find $\alpha_4, \alpha_5, \alpha_6$ that get the two rotation matrices to agree

What is $R(\alpha_4, \alpha_5, \alpha_6)$?



$$R(\alpha_4, \alpha_5, \alpha_6) = R_z(\alpha_4) R_x(\alpha_5) R_y(\alpha_6)$$

$$R_z(\alpha_4) = \begin{bmatrix} \cos(\alpha_4) & -\sin(\alpha_4) & 0 \\ \sin(\alpha_4) & \cos(\alpha_4) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R_x(\alpha_5) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\alpha_5) & -\sin(\alpha_5) \\ 0 & \sin(\alpha_5) & \cos(\alpha_5) \end{bmatrix}$$

$$R_y(\alpha_6) = \begin{bmatrix} \cos(\alpha_6) & 0 & \sin(\alpha_6) \\ 0 & 1 & 0 \\ -\sin(\alpha_6) & 0 & \cos(\alpha_6) \end{bmatrix}$$

$$R_z(\alpha_4)\, R_x(\alpha_5) = \begin{bmatrix} \cos(\alpha_4) & -\sin(\alpha_4)\cos(\alpha_5) & \sin(\alpha_4)\sin(\alpha_5) \\ \sin(\alpha_4) & \cos(\alpha_4)\cos(\alpha_5) & -\cos(\alpha_4)\sin(\alpha_5) \\ 0 & \sin(\alpha_5) & \cos(\alpha_5) \end{bmatrix}$$

$$R_z(\alpha_4)R_x(\alpha_5)R_y(\alpha_6) = \left[\begin{array}{c|c|c} \cos(\alpha_4)\cos(\alpha_6) - \sin(\alpha_4)\sin(\alpha_5)\sin(\alpha_6) & -\sin(\alpha_4)\cos(\alpha_5) & \cos(\alpha_4)\sin(\alpha_6) + \sin\alpha_4\sin\alpha_5\sin\alpha_6 \\ \hline \sin\alpha_4\cos\alpha_6 + \cos\alpha_4\sin\alpha_5\sin 6 & \cos(\alpha_4)\cos(\alpha_5) & \sin\alpha_4\sin\alpha_6 - \cos\alpha_4\sin\alpha_5\sin\alpha_6 \\ \hline -\cos\alpha_5\sin\alpha_6 & \sin\alpha_5 & \cos\alpha_5\cos\alpha_6 \end{array}\right]$$

Two cases need to be considered,

1) $\cos(\alpha_5) = 0$, $\sin(\alpha_5) = \pm 1$

2) $\cos(\alpha_5) \neq 0$

Suppose that we wanted to find $\alpha_4, \alpha_5, \alpha_6$ such that $R(\alpha_4, \alpha_5, \alpha_6) = R_z(\alpha_4) \cdot R_x(\alpha_5)\, R_y(\alpha_5) = R^*$

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1) if $R_{12} = R_{22} = R_{31} = R_{33}$, $R_{32} = \pm 1$
then we have $\cos(\alpha_5) = 0$, $\sin(\alpha_5) = \pm 1$
$\Rightarrow \alpha_5 = \pm \frac{\pi}{2}$

$$R(\alpha_4, \alpha_5, \alpha_6) = \begin{bmatrix} c_4 c_6 \mp s_4 s_6 & 0 & c_4 s_6 \pm s_4 c_6 \\ s_4 c_6 \pm c_4 s_6 & 0 & s_4 s_6 \mp c_4 c_6 \\ 0 & \pm 1 & 0 \end{bmatrix}$$

$$\Rightarrow R(\alpha_4, \alpha_5, \alpha_6) = \begin{bmatrix} \cos(\alpha_4 \pm \alpha_6) & 0 & \sin(\alpha_4 \pm \alpha_6) \\ \sin(\alpha_4 \pm \alpha_6) & 0 & -\cos(\alpha_4 \pm \alpha_6) \\ 0 & \pm 1 & 0 \end{bmatrix}$$

$Zxy \Rightarrow$ if $R_{32} = 1 \Rightarrow \alpha_5 = \frac{\pi}{2}$
$\downarrow$
$\pi/2$

$\alpha_4 + \alpha_6 = \text{atan2}(R_{21}, R_{11})$

2) $R_{32} \neq \pm 1$ and one of R_{21} or R_{22} and one of R_{31} or R_{33} is non-zero,
$\alpha_4 = \text{atan2}(-R_{12}, R_{22})$
$\alpha_6 = \text{atan2}(-R_{31}, R_{33})$
$\alpha_5 = \text{asin}(R_{32})$

So, we determined $\alpha^* = (\alpha_1^*, \alpha_2^*, \alpha_3^*, \alpha_4^*, \alpha_5^*, \alpha_6^*)$
such that $g_e(\alpha^*) = g_e^*$

1) Break up forward kinematics into wrist position & hand orientation.
2) Solve for wrist position
3) " " hand orientation.

MANIPULATOR JACOBIAN

- The manipulator Jacobian gives the end-effector velocity as a function of the joint configuration and the joint velocity.

- There's a generic definition available, but we are going to take advantage of the structure of the forward kinematics to compute the Jacobian.
  1) product of Lie groups
  2) " " exponentials.