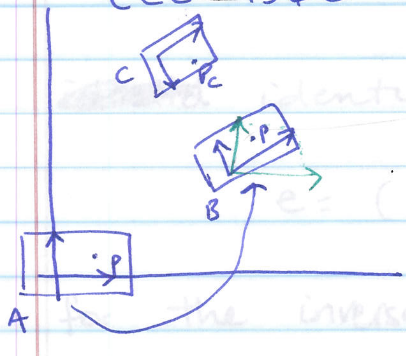
ECE 4560

08/27/2007



when a rigid body experiences a displacement g, the point p undergoes a transformation:

$$g \cdot p = (d, R) \cdot p = \underbrace{d}_{\text{translation (w/o rotation)}} + \underbrace{Rp}_{\text{rotation}}$$

$\cdot$ defines an operation that goes from (d, R) & p to p'

if a rigid body undergoes two displacements g_1 & g_2, then the total displacement g and ~~the~~ the individual displacements are related by:

$$g = g_1 \cdot g_2 = (d_1 R_1) \cdot (d_2 R_2) = (d_1 + R_1 d_2, R_1 R_2)$$

$$P_c^A = g_c^A \cdot p = d_{AC}^A + R_{AC} \cdot p$$

$$P_c^A = d_{AB}^A + R_{AB}^A (d_{BC}^B + R_{BC}^B p)$$

$$= (d_{AB}^A, R_{AB}^A) \cdot (d_{BC}^B, R_{BC}^B) \cdot p$$

order matters!

$(d_2, R_2) \cdot (d_1, R_1) = (d_2 + R_1 d_1, R_2 R_1) \neq$

$(d_1, R_1)(d_2, R_2) = (d_1 + R_1 d_2, R_1 R_2)$

Goal: product structure of displacements?

IDENTITY: $(d, R) \cdot (d_e, R_e) = (d, R)$ ← desired

$(d + R d_e, R R_e) = (d, R)$

$\Rightarrow d + R d_e = d$

$\Rightarrow R\, d_e = 0$

$\Rightarrow R$ is invertible

$\Rightarrow d_e = 0$

$R^{-1}(R R_e = R)$

$\Rightarrow R_e = \mathbb{1}$

$\Rightarrow$

$$\left(g_B^A\right)^{-1} \cdot P_B^A = \begin{Bmatrix} -5\sqrt{2} \\ 0 \end{Bmatrix} + R(-\pi/4) \begin{Bmatrix} 5 + \sqrt{2}/2 \\ 5 + \sqrt{2}/2 \end{Bmatrix} = \begin{Bmatrix} \\ \end{Bmatrix}$$

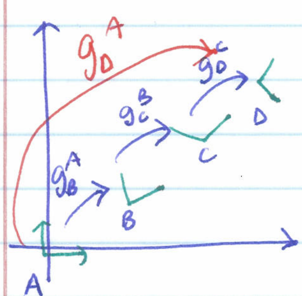
$$= \begin{Bmatrix} -5\sqrt{2} \\ 0 \end{Bmatrix} + \begin{Bmatrix} 5\sqrt{2}/2 + 2/4 + \frac{5\sqrt{2}}{2} + \frac{2}{4} \\ -\frac{5\sqrt{2}}{2} - \frac{2}{4} + \frac{5\sqrt{2}}{2} + \frac{2}{4} \end{Bmatrix}$$

$$= \begin{Bmatrix} 1 \\ 0 \end{Bmatrix}$$

$$\Rightarrow \quad g_B^A = \left(\begin{Bmatrix} 5 \\ 5 \end{Bmatrix}, R(\pi/4) \right)$$

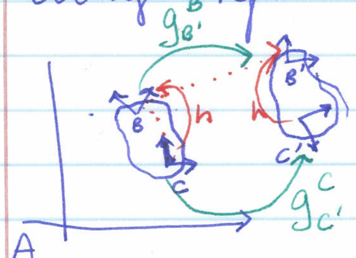
$$\left(g_B^A\right)^{-1} = \left(\begin{Bmatrix} -5\sqrt{2} \\ 0 \end{Bmatrix}, R(-\pi/4) \right\} = g_A^B$$

If order matters, then what is g_D^A



$$g_D^A = g_B^A \, g_C^B \, g_D^C$$

Can we use the operations to understand how to change reference frames of a displacement?



You know: $g_B^{B'}$ (what you've experienced)

h (how your friend has to displace to be in your seat)

What is $g_{C'}^C$? (HINT: follow arrows)

$$g_{C'}^C = h \, g_{B'}^B \, h^{-1}$$

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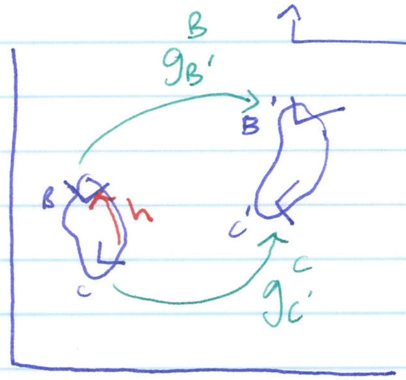
HW: can go online for <u>incomplete</u> Matlab code.

given $g_{B'}^{B}$ and h, then $g_{C'}^{C} = h g_{B'}^{B} h^{-1}$ ← adjoint operation

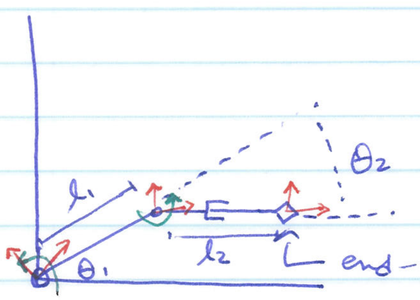
$g_{C'}^{C} = Ad_h\, g_{B'}^{B} = h\, g_{B'}^{B}\, h^{-1}$ ← change of coordinate frame ("perspective")



$Ad_h g = h g h^{-1}$

if A invertible, then there exists a Λ and a T such that $A = T^{-1}\Lambda T$ / $\Lambda = TAT^{-1}$

Can apply these ideas to manipulators:



what is the end-effector's configuration?

$$g_e = \begin{Bmatrix} l_1 \cos\theta_1 + l_2 \cos(\theta_1 + \theta_2) \\ l_1 \sin\theta_1 + l_2 \sin(\theta_1 + \theta_2) \\ \theta_1 + \theta_2 \end{Bmatrix} \Rightarrow \text{vector form} \begin{Bmatrix} x \\ y \\ \theta \end{Bmatrix}$$

There's a more programmatic way to do this.

$(\vec{d}, R)$

$$g_1^0 = \left(\begin{Bmatrix} 0 \\ 0 \end{Bmatrix}, R(\theta_1) \right)$$

$$g_2^1 = \left(\begin{Bmatrix} l_1 \\ 0 \end{Bmatrix}, R(\theta_2) \right)$$

$$g_3^2 = \left(\begin{Bmatrix} l_2 \\ 0 \end{Bmatrix}, \mathbb{1} \right)$$

→

$$g_e = g_1^0 g_2^1 g_3^2 = (\vec{d}_1, R_1)\cdot(\vec{d}_2, R_2)\cdot(\vec{d}_3, R_3)$$
$$= (0, R_1)\cdot(\vec{d}_2, R_2)\cdot(\vec{d}_3, \mathbb{1})$$
$$= (R_1\vec{d}_2, R_1R_2)\cdot(d_3, \mathbb{1})$$
$$= (R_1\vec{d}_2 + R_1R_2\vec{d}_3, R_1R_2)$$
$$= \left(\begin{Bmatrix} l_1\cos(\theta_1) + l_2\cos(\theta_1+\theta_2) \\ l_1\sin(\theta_1) + l_2\sin(\theta_1+\theta_2)\end{Bmatrix}, R(\theta_1+\theta_2)\right)$$

The space of planar rigid body configurations/transformations is called SE(2)

↳ Special Euclidean

It is an instance of a Lie group.

Definition: A *group* is a mathematical structure consisting of a set of elements that can be indexed (set can be finite/infinite, continuous/discrete). Let $G = \{g_j\}$, $j = I$ (the index set), denote this set of elements.

The group has:

1) an associative group, denoted by $*$, called the group product.
2) a unique element e called the identity element such that
$$e * g_j = g_j * e = g_j \quad \text{for all } g_j \in G$$
3) for every $g_j \in G$, an inverse element g_j^{-1}, called the *inverse*, such that $g_j * g_j^{-1} = g_j^{-1} * g_j = e$

Definition: A Lie group is a group G which is also a smooth manifold for which the group product and inverse are smooth.

i.e. infinitely differentiable

no unique way to represent G

$SE(2)$ $\begin{Bmatrix} x \\ y \\ \theta \end{Bmatrix}$ $(\vec{d}, R)$

Homogenous Coordinates

$(\vec{d}, R) \longrightarrow \left[\begin{array}{c|c} R & \vec{d} \\ \hline 0 & 1 \end{array}\right]$ $\quad R: 2\times 2$, $\quad d: 2\times 1$

$\begin{array}{cc} R & d \\ \left[\begin{array}{cc|c} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot \end{array}\right] \end{array}$

so, $e = \left[\begin{array}{c|c} \mathbb{1} & 0 \\ \hline 0 & 1 \end{array}\right]$

$$g_1 g_2 = \begin{bmatrix} R_1 & d_1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} R_2 & d_2 \\ 0 & 1 \end{bmatrix} = \left[\begin{array}{c|c} R_1 R_2 & d_1 + R_1 d_2 \\ \hline 0 & 1 \end{array}\right]$$

$$g^{-1} = \left[\begin{array}{c|c} R^{-1} & -R^{-1} d \\ \hline 0 & 1 \end{array}\right]$$

what about points?

now represented by $\begin{bmatrix} \vec{p} \\ 1 \end{bmatrix} = \begin{Bmatrix} x \\ y \\ 1 \end{Bmatrix}$

$$g \circ q = \left[\begin{array}{c|c} R & d \\ \hline 0 & 1 \end{array}\right] \begin{bmatrix} p \\ 1 \end{bmatrix} = \begin{bmatrix} Rp + d \\ 1 \end{bmatrix}$$

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Homogenous Coordinates

Now, want to write g as and points as

$$g = \begin{bmatrix} R & d \\ 0 & 1 \end{bmatrix} \qquad q = \begin{bmatrix} p \\ 1 \end{bmatrix}$$

Action g on a point q is

$$g \cdot q = \begin{bmatrix} R & d \\ 0 & 1 \end{bmatrix} \begin{bmatrix} p \\ 1 \end{bmatrix} = \begin{bmatrix} Rp + d \\ 1 \end{bmatrix}$$

Given two points q_2 and q_1, I can compute a vector via subtraction:

$$w = q_2 - q_1 = \begin{bmatrix} p_2 \\ 1 \end{bmatrix} - \begin{bmatrix} p_1 \\ 1 \end{bmatrix} = \begin{bmatrix} p_2 - p_1 \\ 0 \end{bmatrix}$$

Conversly, a trajectory can be generated from q_1 to q_2 using w

$q(t) = q_1 + wt \quad \leftarrow$ associated with differential equation $\dot{q} = w$, $q(0) = q_1$

$q(0) = q_1$ and $q(1) = q_2$

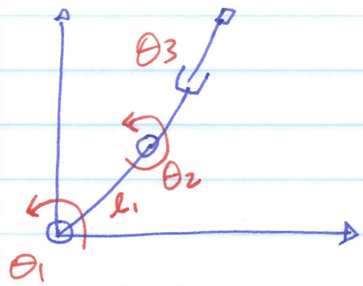
Another reason for distinction is the action of g on w:

$$g \cdot w = \begin{bmatrix} R & d \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} u \\ 0 \end{bmatrix} = \begin{bmatrix} Ru \\ 0 \end{bmatrix}$$

MANIPULATORS AND SE(2)!

Let's consider a manipulator task:

θ_3 is a length



Recall configuration of end effector

$$g_e = g_1^0 g_2^1 g_e^2 = \begin{bmatrix} R(\theta_1) & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} R(\theta_2) & d \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & d_2 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} R(\theta_1+\theta_2) & R(\theta_1)d_1 + R(\theta_1+\theta_2)d_2 \\ 0 & 1 \end{bmatrix}$$

$$d_1 = \begin{Bmatrix} l_1 \\ 0 \end{Bmatrix} \qquad d_2 = \begin{Bmatrix} \theta_3 \\ 0 \end{Bmatrix}$$

What is the end effector config. for $\Theta = (\pi/6, \pi/6, 1)$?
$l_1 = 1$ $\qquad \theta_1 \quad \theta_2 \quad \theta_3$

$$g_e = \begin{bmatrix} R(\pi/6 + \pi/6) & R(\pi/6)d_1 + R(\pi/6 + \pi/6)d_2 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} R(\pi/3) & \begin{Bmatrix} \frac{1+\sqrt{3}}{2} \\ \frac{1+\sqrt{3}}{2} \end{Bmatrix} \\ 0 & 1 \end{bmatrix} = g_A^o$$

$$R(\pi/6)d_1 = \begin{bmatrix} \sqrt{3}/2 & -\frac{1}{2} \\ 1/2 & \sqrt{3}/2 \end{bmatrix} \begin{Bmatrix} l_1 \\ 0 \end{Bmatrix} = \begin{Bmatrix} \frac{\sqrt{3}l_1}{2} \\ \frac{l_1}{2} \end{Bmatrix}$$

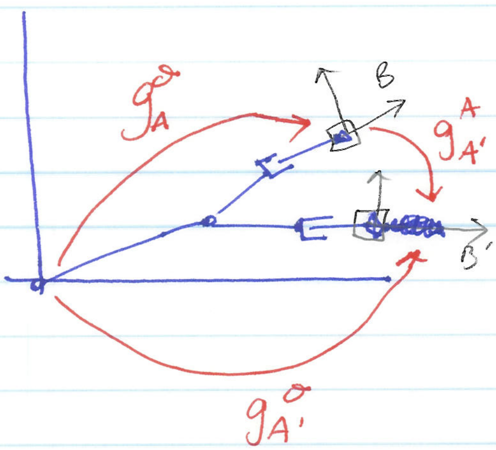
$$R(\pi/3)d_2 = \begin{bmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{bmatrix} \begin{pmatrix} \theta_3 \\ 0 \end{pmatrix} = \begin{bmatrix} \frac{\theta_3}{2} \\ \frac{\sqrt{3}\theta_3}{2} \end{bmatrix}$$

$$= \begin{pmatrix} \frac{1+\sqrt{3}}{2} \\ \frac{1+\sqrt{3}}{2} \end{pmatrix}$$

So, then it grabs something and move to $\Theta = (\pi/6, -\pi/6, 2)$. Where is it?

$$g_e = \left[\begin{array}{c|c} R(0) & R(\pi/6)d_1 + R(0)d_2 \\ 0 & 1 \end{array}\right]$$

$$= \left[\begin{array}{c|c} \mathbb{1} & \begin{Bmatrix} 2+\sqrt{3}/2 \\ 1/2 \end{Bmatrix} \\ \hline 0 & 1 \end{array}\right] = g^{o}_{A'}$$



what transformation did the end effector undergo?

$$g^{A}_{A'} = (g^{o}_{A})^{-1} g^{o}_{A}$$

$$g^{A}_{A'} = \left[\begin{array}{c|c} R(-\pi/3 & \begin{Bmatrix} 0 \\ -2\sqrt{3} \end{Bmatrix} \\ \hline 0 & 1 \end{array}\right]$$

$\downarrow$ work

$$g^{A}_{A'} = (g^{o}_{A})^{-1} (g^{o}_{A})$$

$$g^{-1} = \left[\begin{array}{c|c} R^T & -R^T d \\ \hline 0 & 1 \end{array}\right]$$

$$R^{-1} = R^T$$

$$= \left[\begin{array}{c|c} R(\pi/3) & \begin{Bmatrix} \frac{1+\sqrt{3}}{2} \\ \frac{1+\sqrt{3}}{2} \end{Bmatrix} \\ \hline 0 & 1 \end{array}\right]^{-1} \left[\begin{array}{c|c} R(0) & \begin{Bmatrix} 2+\sqrt{3}/2 \\ 1/2 \end{Bmatrix} \\ \hline 0 & 1 \end{array}\right]$$

$$= \left[\begin{array}{c|c} R(-\pi/3) & -R(-\pi/3)\begin{Bmatrix} (1+\sqrt{3})/2 \\ (1+\sqrt{3})/2 \end{Bmatrix} \\ \hline 0 & 1 \end{array}\right] \left[\begin{array}{c|c} \mathbb{1} & \begin{Bmatrix} 2+\sqrt{3}/2 \\ 1/2 \end{Bmatrix} \\ \hline 0 & 1 \end{array}\right]$$

$$= \left[\begin{array}{c|c} R(-\pi/3) & R(-\pi/3)\begin{Bmatrix} 2+\sqrt{3}/2 \\ 1/2 \end{Bmatrix} - R(-\pi/3)\begin{Bmatrix} (1+\sqrt{3})/2 \\ (1+\sqrt{3}/2 \end{Bmatrix} \\ \hline 0 & \end{array}\right]$$

$$g^{A}_{A'} = \left[\begin{array}{c|c} R(-\pi/3) & R(-\pi/3)\begin{Bmatrix} 2+\sqrt{3}/2 - (1+\sqrt{3})/2 \\ 1/2 - (1+\sqrt{3})/2 \end{Bmatrix} \\ \hline 0 & 1 \end{array}\right]$$

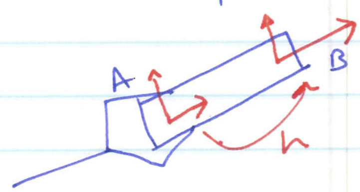
$\hookrightarrow$

$$= \left[\begin{array}{c|c} R(-\pi/3) & R(-\pi/3)\begin{Bmatrix} 3/2 \\ -\sqrt{3}/2 \end{Bmatrix} \\ \hline 0 & 1 \end{array}\right]$$

$$= \left[\begin{array}{c|c} R(-\pi/3) & \begin{Bmatrix} 0 \\ -2\sqrt{3} \end{Bmatrix} \\ \hline 0 & 1 \end{array}\right]$$

Ok, so now what transformations did the object in the end-effector's grip undergo if the transformation from grip to object is

$$h = \left[\begin{array}{c|c} \mathbb{1} & \begin{Bmatrix} 1/4 \\ 0 \end{Bmatrix} \\ \hline 0 & 1 \end{array}\right]$$



Just follow arrows to get the adjoint:

$$g_{B'}^{B} = Ad_{h^{-1}}\, g_{A'}^{A} = h^{-1} g_{A'}^{A} h$$

Adjoint defined: $Ad_h\, g = h g h^{-1}$

$$g_{B'}^{B} = \left[\begin{array}{c|c} \mathbb{1} & d_h \\ \hline 0 & 1 \end{array}\right]^{-1} \left[\begin{array}{c|c} R(-\pi/3) & \begin{Bmatrix} 0 \\ -2\sqrt{3} \end{Bmatrix} \\ \hline 0 & 1 \end{array}\right] \left[\begin{array}{c|c} \mathbb{1} & d_h \\ \hline 0 & 1 \end{array}\right]$$

$$= \left[\begin{array}{c|c} \mathbb{1} & -d_h \\ \hline 0 & 1 \end{array}\right] \left[\begin{array}{c|c} R(-\pi/3) & \begin{Bmatrix} 0 \\ -2\sqrt{3} \end{Bmatrix} + R(-\pi/3) d_h \\ \hline 0 & 1 \end{array}\right]$$

$$R(-\pi/3) d_h = \begin{bmatrix} 1/2 & \sqrt{3}/2 \\ -\frac{\sqrt{3}}{2} & 1/2 \end{bmatrix} \begin{Bmatrix} 1/4 \\ 0 \end{Bmatrix}$$

$$= \begin{Bmatrix} 1/8 \\ -\sqrt{3}/8 \end{Bmatrix}$$

$\longrightarrow$

$$= \left[\begin{array}{c|c} \mathbb{1} & -d_h \\ \hline 0 & 1 \end{array}\right] \left[\begin{array}{c|c} R(-\pi/3) & \begin{Bmatrix} 1/8 \\ -(2+1/8)\sqrt{3} \end{Bmatrix} \\ \hline 0 & 1 \end{array}\right]$$

$$= \left[\begin{array}{c|c} R(-\pi/3) & \begin{Bmatrix} -1/8 \\ -(2+1/8)\sqrt{3} \end{Bmatrix} \\ \hline 0 & 1 \end{array}\right]$$