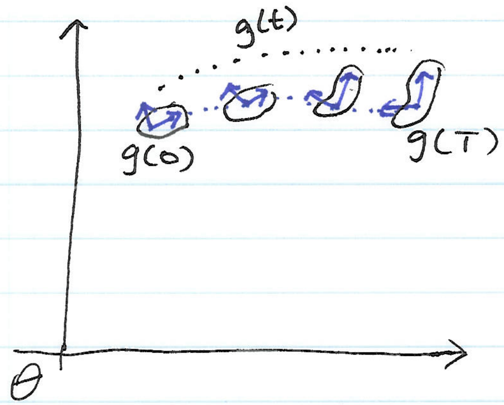
ECE 4560 08/22/07

Planar kinematics
- definitions
- coordinates and transformations
- reference frame.

Definition: • A <u>rigid body</u> is a collection of points/particles which have a *fixed* relationship amongst themselves.

• A <u>rigid body motion</u> describes ~~the~~ how the individual particles of a rigid body move as a function of time. Equivalently, a rigid body motion is the motion of the body fixed reference frame.



g(t) - describes the rigid body motion.

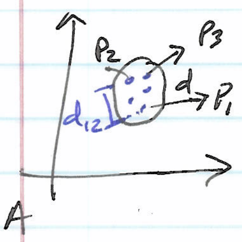
Def: • A <u>displacement</u> / <u>transformation</u> is the movement or motion of a rigid ~~body~~ w/o reference to time.
  • from g(0) to g(T) is a displacement

• A <u>planar rigid body motion</u> is a rigid body motion during which all of the particles remain in a plane (or a set of parallel planes)

• The <u>degrees of freedom</u> (DoF) of a motion is the minimal number of independent variables needed to uniquely specify the motion of an object.
  - Why important? : 'cuz the variables we'll use

sometimes have a bigger dimension than the dof.

Q: How many dof in planar motion?



1) location of particle P_i denoted by (x_i, y_i)

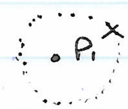
2) Consider P_1. It has 2 dof since $d = (d_x, d_y)$ is arbitrary.

3) Consider P_2. It must satisfy some fixed relationship w/ respect to P_1, by definition of a rigid body.

d_{12} = distance b/w P_1 and P_2.

$\Rightarrow$ i.e, fixed distance

$$(x_2 - x_1)^2 + (y_2 - y_1)^2 = d_{12}^2$$



we started w/ 2 DOF for P_2 but we added one constraint $\Rightarrow$ 2 DOF − 1 constraint = 1 DOF.

4) Consider P_3. 2 DOF − 2 constraints = 0 DOF

$$(x_3 - x_2)^2 + (y_3 - y_2)^2 = d_{23}^2$$
$$(x_3 - x_1)^2 + (y_3 - y_1)^2 = d_{13}^2$$

5) Consider P_i, $i > 3$ 2 DOF, $i-1$ constraints

$\Rightarrow$ 2 real constraints, $i-3$ redundant ones

$\Rightarrow$ 0 DOF.

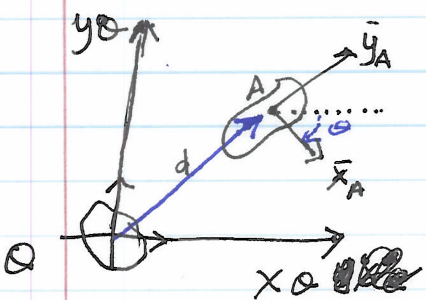
meaning: If I want to describe the configuration associated to a planar rigid body, what do I need?

$\Rightarrow (x, y, \theta)$

COORDINATES:

The 3 DOF for planar objects are position and orientation.

(position → configuration)



configuration of a rigid body:

$$g = (x, y, \theta)$$
$$g = (\vec{d}, \theta)$$
$$g = (\vec{d}, R(\theta))$$
$$g = (x, y, R(\theta))$$

$$R(\theta) = \begin{bmatrix} \cos(\theta) & -\sin\theta \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

$$R(\theta) \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} = \begin{Bmatrix} \cos\theta \\ \sin\theta \end{Bmatrix}$$

$$R(\theta)\, R^T(\theta) = \mathbb{1}$$
$$R^{-1}(\theta) = R^T(\theta)$$

$$\begin{bmatrix} - e_1 - \\ - e_2 - \end{bmatrix} \begin{bmatrix} | & | \\ e_1 & e_2 \\ | & | \end{bmatrix} = \begin{bmatrix} e_1 \cdot e_1 & e_1 \cdot e_2 \\ e_2 \cdot e_1 & e_2 \cdot e_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$\Rightarrow$ 3 constraints:

$e_1 \cdot e_1 = 1$

$e_1 \cdot e_2 = 0$

$e_2 \cdot e_2 = 1$

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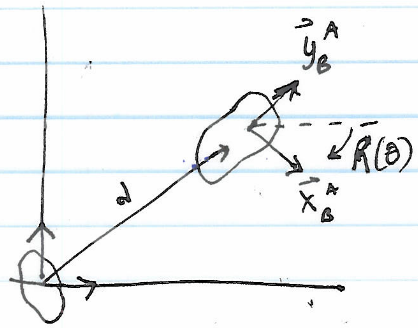
COORDINATES



The 3 DOF for planar objects are
position + orientation
configuration

configuration of a rigid body:

$$g = (x, y, \theta)^T \quad \text{(vector notation)}$$

or equivalently $(\vec{d}, R)$

$$R = R(\theta) = \begin{array}{cc} \vec{x} & \vec{y} \\ \left[\begin{array}{c} \cos\theta \\ \sin\theta \end{array}\right. & \left.\begin{array}{c} -\sin\theta \\ \cos\theta \end{array}\right] \end{array}$$

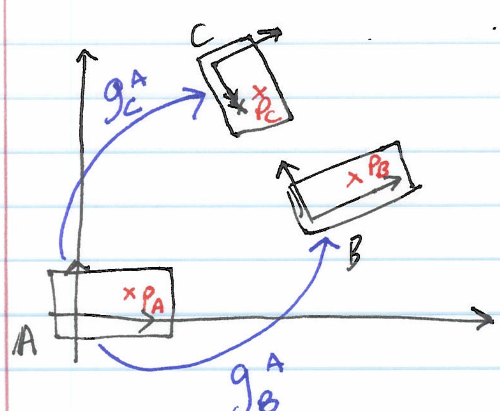
to go b/w θ & R:

$$R = \begin{pmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{pmatrix} \qquad \theta = \text{atan2}(R_{21}, R_{11})$$

matlab fn: help atan2

In a sense, the coordinates also describe a displacement / transformation – in above figure, from origin to new location.

Let's consider another scenario:



in the coordinate frame of the rigid body, the point p is located at

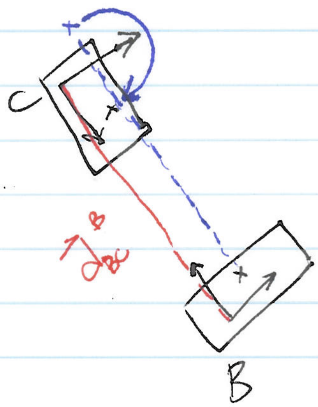
$$p = \alpha_1 \vec{x} + \alpha_2 \vec{y} = \begin{Bmatrix} \alpha_1 \\ \alpha_2 \end{Bmatrix}$$

for our two cases:

$$P_B^A = \vec{d}_{AB}^{\,A} + \alpha_1 \vec{x}_B^{\,A} + \alpha_2 \vec{y}_B^{\,A} = \vec{d}_{AB}^{\,A} + \begin{Bmatrix} \alpha_1 \\ \alpha_2 \end{Bmatrix}_B^A$$

$$P_C^A = \vec{d}_{AC}^{\,A} + \alpha_1 \vec{x}_C^{\,A} + \alpha_2 \vec{y}_C^{\,A} = \vec{d}_{AC}^{\,A} + \begin{Bmatrix} \alpha_1 \\ \alpha_2 \end{Bmatrix}_C^A$$

where is the point P_C located in B's reference frame?
(i.e. what is P_C^B?)



$$P_C^B = \vec{d}_{BC}^{\,B} + \vec{p}\,'$$

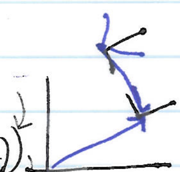
↳ effect of the rotation.

we know that P_C is located at:

$$\vec{P}_C^{\,A} = \vec{d}_{AC}^{\,A} + \alpha_1 \vec{x}_C^{\,A} + \alpha_2 \vec{y}_C^{\,A}$$

if we consider looking at it from B's coordinate system wrt A's frame:

$$\vec{P}_C^{\,A} = \vec{d}_{AB}^{\,A} + \vec{d}_{BC}^{\,B} + \alpha_1 \left(\vec{x}_B^{\,A} \cos(\theta_{BC}) + \vec{y}_B^{\,A} \sin(\theta_{BC}) \right) + \alpha_2 \left(-\vec{x}_B^{\,A} \sin(\theta_{BC}) + \vec{y}_B^{\,A} \cos(\theta_{BC}) \right)$$



this is p' written as a function of p and θ_{BC}

$$P_C^A = \vec{d}_{AB}^{\,A} + \vec{d}_{BC}^{\,B} + (\alpha_1 \cos(\theta_{BC}) - \alpha_2 \sin(\theta_{BC}))\vec{x}_B^{\,A} + (\alpha_1 \sin(\theta_{BC}) + \alpha_2 \cos(\theta_{BC}))\vec{y}_B^{\,A}$$

$$P_C^A = \vec{d}_{AB}^{\,A} + \vec{d}_{BC}^{\,B} + R(\theta_{BC}) \begin{Bmatrix} \alpha_1 \\ \alpha_2 \end{Bmatrix}_B^A$$

this should be in A's frame!

if $\vec{d}_{BC}^{\,B} = \begin{Bmatrix} d_1 \\ d_2 \end{Bmatrix}_C^B$, then in terms of A's frame:

$$\vec{d}_{BC}^{\,A} = R(\theta_{AB}) \begin{Bmatrix} d_1 \\ d_2 \end{Bmatrix}_C^B = R(\theta_{AB})\, \vec{d}_{BC}^{\,A}$$

$$\vec{P}_c^{\,A} = \vec{d}_{AB}^{\,A} + R(\theta_{AB})\, \vec{d}_{BC}^{\,B} + R(\theta_{BC}) \begin{Bmatrix} \alpha_1 \\ \alpha_2 \end{Bmatrix}_B^A$$

now, originally the point located at body/frame B was:

$$\cancel{\vec{P}_B^{\,A} = \vec{d}_{AB}^{\,A} + R(\theta_{BB})}$$

$$\vec{P}_B^{\,A} = \vec{d}_{AB}^{\,A} + \begin{Bmatrix} \alpha_1 \\ \alpha_2 \end{Bmatrix}_B^A = \vec{d}_{AB}^{\,A} + R(\theta_{AB}) \begin{Bmatrix} \alpha_1 \\ \alpha_2 \end{Bmatrix}_A^A$$

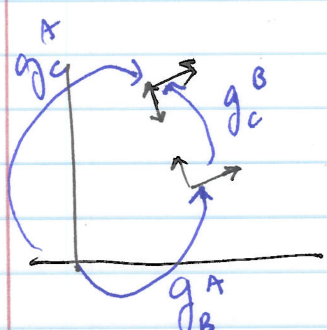
going back to P_c^A

$$\vec{P}_c^{\,A} = \vec{d}_{AB}^{\,A} + R(\theta_{AB})\, \vec{d}_{BC}^{\,B} + R(\theta_{AB}) R(\theta_{BC}) \begin{Bmatrix} \alpha_1 \\ \alpha_2 \end{Bmatrix}_A^A$$

$$= \vec{d}_{AB}^{\,A} + R(\theta_{AB}) \left[\vec{d}_{BC}^{\,B} + R(\theta_{BC}) \begin{Bmatrix} \alpha_1 \\ \alpha_2 \end{Bmatrix} \right]$$

So, what is $\vec{P}_c^{\,B}$?

$$\vec{P}_c^{\,B} = \vec{d}_{BC}^{\,B} + R(\theta_{BC}) \begin{Bmatrix} \alpha_1 \\ \alpha_2 \end{Bmatrix}$$ ← This is HOW POINTS TRANSFORM



$$(d_c^A, R_c^A) = (d_B^A, R_B^A) \times (d_c^B, R_c^B)$$