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29-1

Example Continued from Friday (lecture 28)

$$J_1^b(\alpha) = \begin{Bmatrix} 0 \\ 0 \\ \hline 1 \end{Bmatrix} \quad J_2^b(\alpha) = \begin{Bmatrix} 0 \\ 0 \\ \hline 1 \end{Bmatrix} \quad J_3^b(\alpha) = \begin{Bmatrix} 0 \\ 0 \\ \hline 1 \end{Bmatrix} \quad J_4^b(\alpha) = \begin{Bmatrix} 1 \\ 0 \\ \hline 0 \end{Bmatrix}$$

We had the following formula for the complete body ~~Jacobian~~ Jacobian.

$$J^b(\alpha) = \left[\begin{array}{c|c|c|c} Ad^{-1}_{g_2 g_3 g_4} J_1^b & Ad^{-1}_{g_3 g_4} J_2^b & Ad^{-1}_{g_4} J_3^b & J_4^b \end{array} \right]$$

where,

$$Ad^{-1}_{g_2 g_3 g_4} = Ad_{(g_2 g_3 g_4)^{-1}} = \left[\begin{array}{c|c} R(-\alpha_2-\alpha_3) & \begin{matrix} l_1 \sin(\alpha_2+\alpha_3) + l_2 \sin\alpha_3 \\ l_1 \cos(\alpha_2+\alpha_3) + l_2 \cos\alpha_3 + \alpha_4 \end{matrix} \\ \hline 0 & 1 \end{array} \right]$$

$$g_2 g_3 g_4 = \left[\begin{array}{c|c} R(\alpha_2) & \begin{Bmatrix} l_1 \\ 0 \end{Bmatrix} \\ \hline 0 & 1 \end{array} \right] \left[\begin{array}{c|c} R(\alpha_3) & \begin{Bmatrix} l_2 \\ 0 \end{Bmatrix} \\ \hline 0 & 1 \end{array} \right] \left[\begin{array}{c|c} 1 & \begin{Bmatrix} \alpha_4 \\ 0 \end{Bmatrix} \\ \hline 0 & 1 \end{array} \right]$$

$$= \left[\begin{array}{c|c} R(\alpha_2+\alpha_3) & \begin{Bmatrix} l_1 \\ 0 \end{Bmatrix} + R(\alpha_2) \begin{Bmatrix} l_2 \\ 0 \end{Bmatrix} + R(\alpha_2+\alpha_3) \begin{Bmatrix} \alpha_4 \\ 0 \end{Bmatrix} \\ \hline 0 & 1 \end{array} \right]$$

$$\Rightarrow (g_2 g_3 g_4)^{-1} = \left[\begin{array}{c|c} R(-\alpha_2-\alpha_3) & -R^T(\alpha_2+\alpha_3) \begin{Bmatrix} l_1 \\ 0 \end{Bmatrix} - R^T(\alpha_3) \begin{Bmatrix} l_2 \\ 0 \end{Bmatrix} - \begin{Bmatrix} \alpha_4 \\ 0 \end{Bmatrix} \\ \hline 0 & 1 \end{array} \right]$$

Recall that for the vector form of SE(2), the adjoint can be written

$$Ad_g \xi = Ad_g \begin{Bmatrix} v \\ \omega \end{Bmatrix} = \left[\begin{array}{c|c} R(\theta) & JT d \\ \hline 0 & 1 \end{array} \right] \begin{Bmatrix} v \\ \omega \end{Bmatrix}$$

2. Product of Exponentials

In the case, $g_e(\alpha) = e^{\xi_1 \alpha_1} \cdots e^{\xi_m \alpha_m} g_0$

$\uparrow$ constant (reference) configuration.

we have the following individual ~~comp~~ body components:

$$J_i^b(\alpha_i) = \left(e^{\xi_i \alpha_i}\right)^{-1} \frac{d}{d\alpha_i}\left(e^{\xi_i \alpha_i}\right) = e^{-\xi_i \alpha_i} e^{\xi_i \alpha_i} \cdot \xi_i = \xi_i$$

for the spatial version, we pretty much get the same:

$$J_i^s(\alpha_i) = \left(\frac{d}{d\alpha_i} e^{\xi_i \alpha_i}\right)\left(e^{-\xi_i \alpha_i}\right)$$

$\hookrightarrow$ switch to homogenous:

$$\left(e^{\hat{\xi}_i \alpha_i}\, \hat{\xi}_i\, e^{-\hat{\xi}_i \alpha_i}\right)$$

$\hookrightarrow$

$$J_i^s(\alpha_i) = Ad_{\left(e^{\hat{\xi}_i \alpha_i}\right)} \hat{\xi}_i \Rightarrow Ad_{e^{\xi_i \alpha_i}} \xi_i$$

$\hookrightarrow$ back to vector form.

$$\Rightarrow J^b(\alpha) = \left[Ad^{-1}_{\left[\left(\prod_{j=2}^{m} e^{\xi_j \alpha_j}\right) \cdot g_0\right]} J_1^b(\alpha_1) \;\cdots\; Ad^{-1}_{\left[\left(\prod_{j=k+1}^{m} e^{\xi_j \alpha_j}\right) g_0\right]} J_k^b(\alpha_k) \;\cdots\; Ad^{-1}_{g_0} J_m^b(\alpha_m) \right]$$

added on 10/31/07:

$$\Rightarrow J^s(\alpha) = \left[J_1^s(\alpha) \quad Ad_{e^{\xi_1 \alpha_1}} J_2^s(\alpha_2) \;\cdots\; Ad_{\left(\prod_{j=1}^{k-1} e^{\xi_j \alpha_j}\right)} J_k^s(\alpha_k) \;\cdots\; Ad_{\left(\prod_{j=1}^{m-1} e^{\xi_j \alpha_j}\right)} J_m^s(\alpha_m) \right]$$

for the spatial Jacobian.

Manipulator Types (based on Workspace)

more functional, less geometric

let n = dimension of task space
m = dimension of joint space

The three (functional) classes to divide manipulators into are:

1) kinematically insufficient $(n > m)$
   - can get a "better" manipulator so that $m = n$
   - can reduce dimension of task space
     $\Rightarrow$ redifining what the task is.

2) kinematically redundant $(n < m)$
   - have more degrees of freedom than needed
     $\Rightarrow$ inverse kinematics have ∞ solutions
   - it's tricky, but it can be dealt with.

3) kinematically sufficient $(n = m)$
   - need to ensure operation in dextrous workspace for maximal mobility / control.
   - equivalently, need to be away from singularities.

Will discuss now.

Definition: A <u>singular configuration</u> or <u>singularity</u> is a joint configuration of an open-chain manipulator in which the end-effector instantaneously loses a degree of freedom of its motion capability versus the number of degrees of freedom that normally prevail (i.e. the end-effector loses its ability to move in one or more directions).

- open-chain means that joint-link ... combinations form a loop.

Practical implications:

1) manipulator loses effectiveness.
2) high joint velocities may be needed near a singular configuration in order to track a specified trajectory.
3) the manipulator can have high mechanical advantage in a singular configuration.
   e.g.: benchpressing in a gym as your arm is fully extended.

Finding singular configurations:

⇒ to find them, examine the manipulator Jacobian.

1) Numerical Approach

$$\dot{\xi} = J(\alpha)\,\dot{\alpha} \quad \Leftarrow \text{ test rank of } J.$$

(# of linearly independant columns)

2) for kinematically sufficient arms

$$\det(J(\alpha)) = 0$$

at a singular configuration.

3) for redundant arms $(n < m)$,

added 11/02/07

$J(\alpha)\,J^T(\alpha)$ has the same rank as $J(\alpha)$ but is square. That means a perfectly valid test is:

$$\det(J(\alpha)\,J^T(\alpha)) = 0$$

for singular configurations.

A Mech. Eng. prof has "issues" with this point.

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4) for kinematically insufficient arms $J^T(\alpha)J(\alpha)$ has the same rank as $J(\alpha)$ but is square. A test for singular configurations is

$$\det(J^T(\alpha)J(\alpha)) = 0.$$

* it is in green, because one should not really even have this case. if an arm is insufficient, it is best to get a new arm or modify task to be sufficient.
* the product $J^T(\alpha)J(\alpha)$ has nonsense units if dimensional analysis is performed.

To find the singular configurations means applying one of the given tests for all α in the joint space and keeping track of the singular ones.

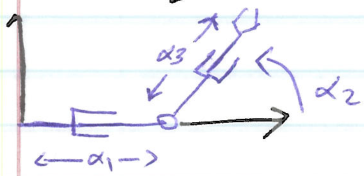
There is a connection b/w loss of rank in the Jacobian and loss of control.

There are usually two occassions when a singularity occurs.

1) Workspace - boundary singularities
   - when manipulator is fully extended or folded back on itself in such a way that the end-effector is at, or near, the workspace boundary.
2) Workspace - interior singularities.
   - Occur away from workspace boundary, generally caused by a lining up of two or more axes.

Example:

$$g_1 = \left[\begin{array}{c|c} \mathbb{1} & \begin{Bmatrix} \alpha_1 \\ 0 \end{Bmatrix} \\ \hline 0 & 1 \end{array}\right] \quad g_2 = \left[\begin{array}{c|c} R(\alpha_2) & \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \\ \hline 0 & 1 \end{array}\right] \quad g_3 = \left[\begin{array}{c|c} \mathbb{1} & \begin{Bmatrix} \alpha_3 \\ 0 \end{Bmatrix} \\ \hline 0 & 1 \end{array}\right]$$



$$g_e(\alpha) = g_1(\alpha_1)\, g_2(\alpha_2)\, g_3(\alpha_3) = \left[\begin{array}{c|c} R(\alpha_2) & \begin{Bmatrix} \alpha_1 + \alpha_3 \cos(\alpha_2) \\ \alpha_3 \sin(\alpha_2) \end{Bmatrix} \\ \hline 0 & 1 \end{array}\right]$$

To compute body manipulator Jacobian, will want individual ones for each joint

$$J_1^b(\alpha_1) = \left(g_1^{-1} \frac{\partial g_1}{\partial \alpha_1}\right)^{\vee} = \begin{Bmatrix} 1 \\ 0 \\ \hline 0 \end{Bmatrix} \qquad J_2^b(\alpha) = \left(g_2^{-1} \frac{\partial g_2}{\partial \alpha_2}\right)^{\vee} = \begin{Bmatrix} 0 \\ 0 \\ \hline 1 \end{Bmatrix}$$

$$J_3^b(\alpha_3) = \left(g_3^{-1} \frac{\partial g_3}{\partial \alpha_3}\right)^{\vee} = \begin{Bmatrix} 1 \\ 0 \\ \hline 0 \end{Bmatrix}$$

Next step is adjoints:

$$\mathrm{Ad}^{-1}_{(g_2 g_3)} J_1^b = \begin{Bmatrix} \cos(\alpha_2) \\ -\sin(\alpha_2) \\ 0 \end{Bmatrix}, \quad \mathrm{Ad}^{-1}_{g_3} J_2^b = \begin{Bmatrix} 0 \\ \alpha_3 \\ \hline 1 \end{Bmatrix}$$

of J_i^b for vector form

$\boxed{\text{where } \mathrm{Ad}_g = \left[\begin{array}{c|c} R & J' d \\ \hline 0 & 1 \end{array}\right]}$

$$\Rightarrow$$

$$J(\alpha) = \begin{bmatrix} | & | & | \\ \mathrm{Ad}^{-1}_{(g_2 g_3)} J_1^b(\alpha_1) & \mathrm{Ad}^{-1}_{g_3} J_2^b(\alpha_2) & J_3^b(\alpha_3) \\ | & | & | \end{bmatrix}$$

$$= \begin{bmatrix} \cos(\alpha_2) & 0 & 1 \\ -\sin(\alpha_2) & \alpha_3 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Now, let's evaluate the body manipulator Jacobian for two cases:

CASE 1: $\alpha = (1, \pi/2, 1)^T$

$\Rightarrow \quad J^b(\alpha) = \begin{bmatrix} 0 & 0 & 1 \\ -1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$

defined on 30-2

Matrix has full rank
(we have full local control)

CASE 2: $(1, 0, 1)^T$

$\rightarrow \quad J^b(\alpha) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

Matrix is not full rank
(lost control of the system, locally)

more generally, let $\alpha = (1, \alpha_2, 1)^T$, then

$J^b(\alpha) = \begin{bmatrix} \cos(\alpha_2) & 0 & 1 \\ -\sin(\alpha_2) & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

loses rank for $\alpha_2 = \{0, \pi\} + 2\pi k$

$\det(J^b(\alpha)) = -\sin(\alpha)$ $\Leftarrow$ as we approach singularity the determinant approaches zero.