10/01/07

18-1

ECE 4560

continued from last time...

final step:

$g_e(\alpha) = g_1(\alpha)\, g_2(\alpha)\, g_3(\alpha)\, g_4(\alpha)\, g_5$

$$= \left[\begin{array}{ccc|c} \cos(\alpha_1) & -\sin(\alpha_1)\cos(\alpha_2+\alpha_3+\alpha_4) & \sin(\alpha_1)\sin(\alpha_2+\alpha_3+\alpha_4) & \\ \sin(\alpha_1) & \cos(\alpha_1)\cos(\alpha_2+\alpha_3+\alpha_4) & -\cos(\alpha_4)\sin(\alpha_2+\alpha_3+\alpha_4) & \cdots \\ 0 & \sin(\alpha_2+\alpha_3+\alpha_4) & \cos(\alpha_2+\alpha_3+\alpha_4) & \\ \hline 0 & 0 & 0 & \end{array}\right.$$

$$\left.\begin{array}{c} -\sin(\alpha_1)\left[l_1\cos(\alpha_2) + l_2\cos(\alpha_2+\alpha_3) + l_3\cos(\alpha_2+\alpha_3+\alpha_4)\right] \\ \cdots\ \cos(\alpha_1)\left[l_1\cos(\alpha_2) + l_2\cos(\alpha_2+\alpha_3) + l_3\cos(\alpha_2+\alpha_3+\alpha_4)\right] \\ l_0 + l_1\sin(\alpha_2) + l_2\sin(\alpha_2+\alpha_3) + l_3\sin(\alpha_2+\alpha_3+\alpha_4) \\ \hline 1 \end{array}\right]$$

MAIN IDEA: concatenate Lie group configuration change from joint to joint (starting at base & going to end-effector)

Product of Exponentials:

Product of Lie groups gave: $g_e = g_1 g_2 \cdots g_n g_{n+1}$

Is there a way to have $g_e = e^{\xi_1 \alpha_1} \cdots e^{\xi_n \alpha_n} g_{n+1}$?

Yes, but we must be careful!

WRONG IDEA FIRST:

It is <u>not</u> as simple as doing $(\xi_i, \alpha_i) = \ln(g_i(\alpha_i))$

Why?

- because the g's may have constant offsets in them, and we know that $e^{\hat{\xi}_i(0)} = \mathbb{1}$, which effectively does nothing.

Consider trying to find

$$g_e = e^{\hat{\xi}_1 \alpha_1} e^{\hat{\xi}_2 \alpha_2} \cdots e^{\hat{\xi}_n \alpha_n} g_0$$

→ new preferred notation for this part when doing product of exponentials. Why? ...

if $\vec{\alpha} = 0$, then

$$g_e(\vec{\alpha}) = g_e(0) = g_0$$

← g_0 is the reference base to end-effector configuration for the manipulator.

A given ξ_i will then say how the reference configuration changes as its associated joint varies (α_i)

$$g_e(0, \ldots 0, \alpha_i, 0 \ldots, 0) = e^{\hat{\xi}_i} g_0$$

2 questions

1) What are the ξ_i?
2) What do they and their exponents represent?

1) What are the ξ_i ?

Here are three basic types corresponding to the three single degree-of-freedom lower-pair joints:

1) revolute: $\xi_i = \begin{Bmatrix} \hat{q}_i \omega_i \\ \omega_i \end{Bmatrix}$

- ω_i - unit vector aligned w/ rotation axis in base frame.
- q_i - point on the axis of rotation.

2) prismatic: $\xi_i = \begin{Bmatrix} v_i \\ 0 \end{Bmatrix}$

- v_i - unit vector aligned w/ translation axis.

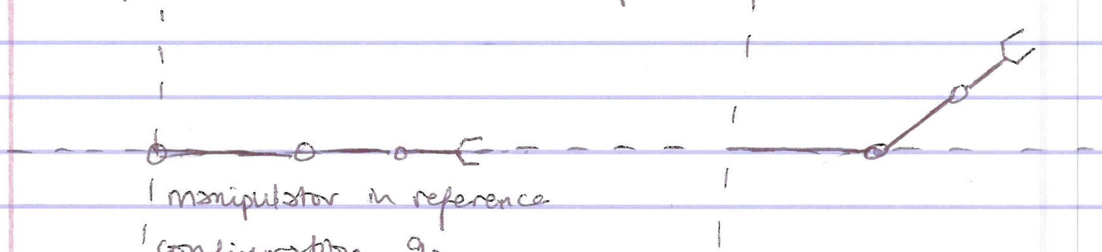
3) helical / screw $\xi_i = \begin{Bmatrix} h\omega_i + \hat{q}_i \omega_i \\ \omega_i \end{Bmatrix}$

- h - pitch of helical motion

but what do they mean / represent for the manipulator?
- they represent vectors / twists in spatial coordinates,
  $\xi_i \in SE(3)$ ← a vector SE(n)

* they give the needed transformation of the base frame (e.g. the spatial frame) so that the transformation of the base frame to the end-effector frame (g_0 - constant) does the right thing.

Example: Planar 3R Manipulator (R: revolute)

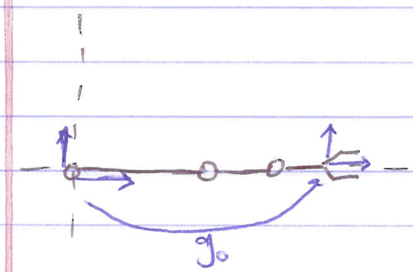


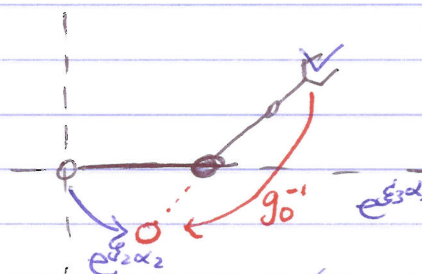
manipulator in reference configuration g_0

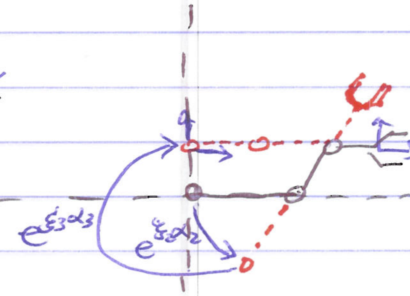
$$g_e = e^{\xi_1 \alpha_1} e^{\xi_2 \alpha_2} e^{\xi_3 \alpha_3} g_0$$

ECE 4560

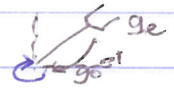
'Ctd from last time...







reference configuration



$e^{\xi_1\alpha_1} \cdots e^{\xi_n\alpha_n} g_0$

$g_e = \cancel{e^{\xi_1\alpha_1}} e^{\xi_2\alpha_2} \cancel{e^{\xi_3\alpha_3}} g_0$

$g_e g_0^{-1} = e^{\xi_2\alpha_2}$

$e^{\xi_2\alpha_2}$ is the necessary "imaginary" motion of the base frame such that transformation under g_0 from this new base frame location takes us to the true g_e.

$e^{\xi_2\alpha_2} e^{\xi_3\alpha_3} g_0$

all of the exponentials move the "imaginary" base frame until it ends up where it should.

Rewind back to ~3 weeks ago: I talked about body vs. spatial interpretations to velocities

relative to imaginary extension of body so that the origin frame is part of the body.

relative to body frame.

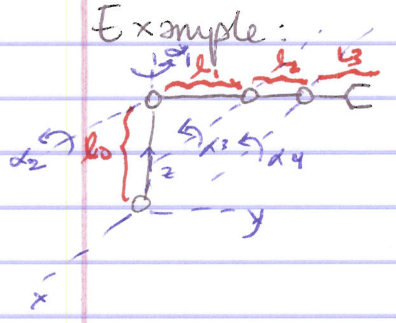
We see that the product of exponentials is a spatial technique for computing the forward kinematics.

Based on seeing how the imaginary base frame changes for each joint combine this effect for all joints to get:

$$g_e(\alpha) = \underbrace{e^{\xi_1\alpha_1} e^{\xi_2\alpha_2} \cdots e^{\xi_n\alpha_n}}_{\text{order matters!}} g_0$$

the (ξ_i, α_i) must enumerate from base to tool frame.

Example:



for product of exponentials forward kinematics, need to find ξ_i & g_0 such that

$$g_e(\alpha) = e^{\hat{\xi}_1 \alpha_1} \cdots e^{\hat{\xi}_4 \alpha_4} g_0$$

$$g_e(\vec{0}) = g_0 = \left[\begin{array}{c|c} \mathbb{1} & \begin{Bmatrix} 0 \\ l_1+l_2+l_3 \\ l_0 \end{Bmatrix} \\ \hline 0 & 1 \end{array}\right]$$

what are ξ_i?

$\boxed{\xi_1}$ rotation axis $\omega_1 = \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix} \Longrightarrow \xi_1 = \begin{Bmatrix} \hat{q}_1 \omega_1 \\ \omega_1 \end{Bmatrix}$

cross product of q_1 w/ ω_1

point on the axis $q_1 = \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix}$ $= \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{Bmatrix}$

q_1 can be any thing on the z axis.

Jacob's choice for any point on axis of rotation as needed by technique.

$\boxed{\xi_2}$ $\omega_2 = \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix}$ rotation axis: x-axis

$q_2 = \begin{Bmatrix} 1 \\ 0 \\ l_0 \end{Bmatrix} \Rightarrow \hat{q}_2 \omega_2 = \begin{Bmatrix} 0 \\ l_0 \\ 0 \end{Bmatrix}$

can be any point on the x axis.

KLIMKA's CHOICE

$$\Rightarrow \xi_2 = \begin{Bmatrix} \hat{q}_2 \omega_2 \\ \omega_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ l_0 \\ 0 \\ 1 \\ 0 \\ 0 \end{Bmatrix}$$

$\boxed{\xi_3}$ $\omega_3 = \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix}$ $q_3 = \begin{Bmatrix} 0 \\ l_1 \\ l_0 \end{Bmatrix}$, $\hat{q}_3 \omega_3 = \begin{Bmatrix} 0 \\ l_0 \\ -l_1 \end{Bmatrix}$

$$\Rightarrow \xi_3 = \begin{Bmatrix} 0 \\ l_0 \\ -l_1 \\ 1 \\ 0 \\ 0 \end{Bmatrix}$$

$\boxed{\xi_4}$ $\omega_4 = \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix}$ $q_4 = \begin{Bmatrix} 0 \\ l_1 + l_2 \\ l_0 \end{Bmatrix}$; $\hat{q}_4 \omega_4 = \begin{Bmatrix} 0 \\ l_0 \\ -l_1 - l_2 \end{Bmatrix}$

$$\Rightarrow \xi_4 = \begin{Bmatrix} \hat{q}_4 \omega_4 \\ \hline \omega_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ l_0 \\ -l_1 - l_2 \\ \hline 1 \\ 0 \\ 0 \end{Bmatrix}$$

Given that ξ_i & g_0, we exponentiate and multiply appropriately,

$$g_e(\alpha) = e^{\xi_1 \alpha_1} e^{\xi_2 \alpha_2} e^{\xi_3 \alpha_4} g_0$$

$$e^{\xi_1 \alpha_1} = \left[\begin{array}{ccc|c} \cos(\alpha_1) & -\sin(\alpha_1) & 0 & 0 \\ \sin(\alpha_1) & \cos(\alpha_1) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right]$$

ECE 4560

10/05/07
20-1

Continued ....

$$g_e(\alpha) = e^{\xi_1 \alpha_1} \cdots e^{\xi_4 \alpha_4} g_0$$

$$e^{\xi_1 \alpha_1} = \left[\begin{array}{ccc|c} \cos(\alpha_1) & -\sin(\alpha_1) & 0 & 0 \\ \sin(\alpha_1) & \cos(\alpha_1) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right]$$

$$e^{\xi_2 \alpha_2} = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & \cos(\alpha_2) & -\sin(\alpha_2) & l_0 \sin(\alpha_2) \\ 0 & \sin(\alpha_2) & \cos(\alpha_2) & l_0(1-\cos(\alpha_2)) \\ \hline 0 & 0 & 0 & 1 \end{array}\right]$$

$$e^{\xi_3 \alpha_3} = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & \cos(\alpha_3) & -\sin(\alpha_3) & l_0 \sin(\alpha_3) + l_1(1-\cos(\alpha_3)) \\ 0 & \sin(\alpha_3) & \cos(\alpha_3) & l_0(1-\cos(\alpha_3)) - l_1(\sin(\alpha_3)) \\ \hline 0 & 0 & 0 & 1 \end{array}\right]$$

after lots of fun trigonometry & algebra (see old lectures)

$$\Rightarrow g_e(\alpha) = \left[\begin{array}{ccc|c} \cos(\alpha_1) & -\sin(\alpha_1)\cos(\alpha_2+\alpha_3+\alpha_4) & \sin(\alpha_1)\sin(\alpha_2+\alpha_3+\alpha_4) & \ldots \\ \sin(\alpha_1) & \cos(\alpha_1)\cos(\alpha_2+\alpha_3+\alpha_4) & -\cos(\alpha_1)\sin(\alpha_2+\alpha_3+\alpha_4) & \\ 0 & \sin(\alpha_2+\alpha_3+\alpha_4) & \cos(\alpha_2+\alpha_3+\alpha_4) & \\ \hline 0 & 0 & 0 & \end{array}\right.$$

$$\ldots \left.\begin{array}{|c} -\sin(\alpha_1)[l_1\sin(\alpha_2) + l_2\cos(\alpha_2+\alpha_3) + l_3\cos(\alpha_2+\alpha_3+\alpha_4)] \\ \cos(\alpha_1)[\, l_1 \quad '' \quad + \quad '' \quad + \quad '' \,] \\ l_0 + l_1\sin(\alpha_2) + l_2\sin(\alpha_2+\alpha_3) + l_3\sin(\alpha_2+\alpha_3+\alpha_4) \\ \hline 1 \end{array}\right]$$

- the final answer agrees with product of Lie groups. (see 18-1)

- in old lectures this example plus another variation of it is worked out (slightly disorganized)

Inverse Kinematics

- determining a feasible joint-configuration given a desired end-effector configuration.
- this is a nonlinear problem.
- clearly, the desired end-effector configuration must be in the workspace to have a solution.
  - sometimes there will be multiple solutions.
- no real general algorithms exist to handle the non-linear inverse kinematics problems.
- the problem is considered solveable if a joint configuration can be determined by an algorithm that finds all joint variables associated w/ a given end-effector configuration
  - * just asking for a function that gives me one solution, if any exist.

two overall techniques:

1. numerical
2. closed-form → algebraic → Pieper's solution when three axes intersect.
   → geometric
   → Paden-Kahan

Main approaches & their attributes:

	Pro	Con
geometric	easy to understand/apply	not systematic not always applicable
algebraic	systematic & general	harder to understand can be harder to use.
Paden-Kahan	easy to understand systematic	not general