ECE 4560 09/05/07

# The Space of Rigid Body Motions

Recall that space of planar rigid body motions described by $SE(2)$

One representation is homogenous coordinates:

$$g = \left[\begin{array}{c|c} R & d \\ \hline 0 & 1 \end{array}\right] \qquad d \in \mathbb{R}^{2\times 1} = E^2$$

$R \in SO(2)$ $(R^T R = \mathbb{1}_{2\times 2})$
↳ special orthogonal space

$\Rightarrow$ composition of $SE(2)$ is $E^2 \times SO(2)$.

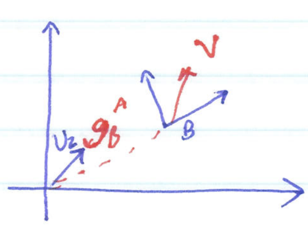
What is $SE(3)$? $E^3 \times SO(3)$
↳ what is this?

$R \in SO(3)$ $\begin{bmatrix} | & | & | \\ e_1 & e_2 & e_3 \\ | & | & | \end{bmatrix}$ $e_3 = e_1 \times e_2$
$R^T R = \mathbb{1}$
right hand rule $\leftarrow \det(R) = +1$

$$g = \begin{bmatrix} R & d \\ 0 & 1 \end{bmatrix} = \left[\begin{array}{c|c} 3\times 3 & 3\times 1 \\ \hline 1\times 3 & 1\times 1 \end{array}\right]$$

## Vectors:

Although vectors have typically been described as disembodied, that is not the case when dealing with rigid body motion.



suppose that $g_B^A = \left[\begin{array}{c|c} R(\pi/4) & \begin{Bmatrix}5\\6\end{Bmatrix} \\ 0 & 1 \end{array}\right]$ and that

$$v^B = \begin{Bmatrix}1\\1\\0\end{Bmatrix}$$

what is $v^A$?

$$v^A = g_B^A \, v^B$$

$$= \left[\begin{array}{c|c} R(\pi/4) & \begin{Bmatrix}5\\6\end{Bmatrix} \\ \hline 0 & 1 \end{array}\right] \begin{Bmatrix}1\\1\\0\end{Bmatrix}$$

$R(\pi/4) = \begin{bmatrix} \sqrt{2}/2 & -\sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix}$

$$= \begin{Bmatrix}0\\\sqrt{2}\\0\end{Bmatrix}$$

Likewise, the vector $v_2^A = \begin{Bmatrix}1\\1\\0\end{Bmatrix}$ appears in frame B

as ....

$$v^B = g_A^B \, v^A = (g_A^B)^{-1} v^A$$

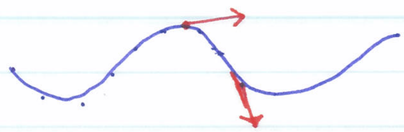
$$= \left[\begin{array}{c|c} R(-\pi/4) & -R(-\pi/4)\,d \\ 0 & 1 \end{array}\right] \begin{bmatrix}1\\1\\0\end{bmatrix} = \begin{Bmatrix}\sqrt{2}\\0\\0\end{Bmatrix}$$

$\Rightarrow$ vectors change frame according to the orientation change of the new frame.

The logical question to ask is: what about vectors in $SE(n)$? and how do they transform?

To answer these questions, we need to go back to first principles.

- vectors arise from infinitesimal displacements associated to trajectories:



curve $p(t) \Rightarrow$ time derivative $\dot{p}(t)$ is a vector. (varies according to the curve)

What is a vector?

1) displacement/change b/w two Euclidean points.

$$v = q_2 - q_1$$

2) mathematical description of the incremental change when following a trajectory in Euclidean space.

Properties of vectors:

1) form a linear space : (can add vectors & multiply by scalar)

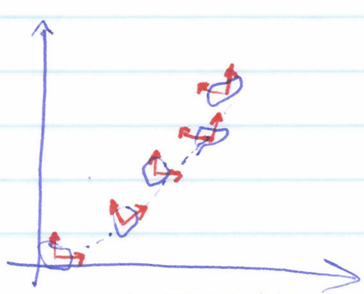
2) transform under reference frame change.

3) there is a mapping from displacements to vectors (and vice-versa)

Velocities in $SE(2)$:

So, let's start with a trajectory in $SE(2)$, given by $g(t)$.

$$g(t) : [t_0, t_1] \to SE(2)$$



Q: What is the velocity of the body moving according to $g(t)$?

$$g(t) = \begin{bmatrix} R(\theta(t)) & d(t) \\ 0 & 1 \end{bmatrix}$$

$$\frac{d}{dt} g(t) = \begin{bmatrix} \frac{d}{dt} R(\theta(t)) & \frac{d}{dt} d(t) \\ 0 & 0 \end{bmatrix}$$

but $\frac{d}{dt} R(\theta(t)) = \begin{bmatrix} -\sin(\theta(t))\dot{\theta}(t) & -\cos(\theta(t))\dot{\theta}(t) \\ \cos(\theta(t))\dot{\theta}(t) & -\sin(\theta(t))\dot{\theta}(t) \end{bmatrix}$

$$= \frac{\partial R(\theta)}{\partial \theta} \cdot \dot{\theta} = DR(\theta) \cdot \dot{\theta}$$

$$DR(\theta) = \begin{bmatrix} -\sin(\theta) & -\cos(\theta) \\ \cos(\theta) & -\sin(\theta) \end{bmatrix}$$

$$\Rightarrow \dot{g}(t) = \begin{bmatrix} DR(\theta(t)) \cdot \dot{\theta}(t) & \dot{d}(t) \\ 0 & 0 \end{bmatrix}$$

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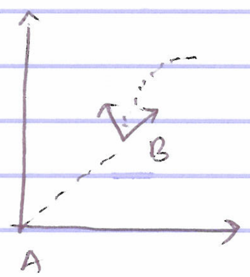
1) MATLAB

- site-license is on webct; just find a copy of program

2) RS-232 → USB adapter comes next week

## Velocities in SE(2) ctd.




$g(t) = g_B^A(t)$

$\frac{d}{dt}$

$$\dot{g}(t) = \left[\begin{array}{c|c} DR(\theta(t))\,\dot{\theta}(t) & \dot{d}(t) \\ \hline 0 & 0 \end{array}\right]$$

has dependence on g.

$$= \dot{g}_B^A(t)$$

I don't like this, so I will ~~transform~~ change description to B's frame.

$$\dot{g}_B^B(t) = g_A^B(t)\,\dot{g}_B^A(t)$$

$$= \left(g_B^A(t)\right)^{-1}\dot{g}_B^A(t)$$

$$= \left[\begin{array}{c|c} R(-\theta(t)) & -R(-\theta(t))d(t) \\ \hline & 1 \end{array}\right]\left[\begin{array}{c|c} DR(\theta(t)) & \dot{d}(t) \\ \hline 0 & 0 \end{array}\right]$$

$R(-\theta)DR(\theta)$

$$= \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} -\sin\theta & -\cos\theta \\ \cos\theta & -\sin\theta \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = J^T \overset{\text{where}}{=} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

ctd

$$= \left[\begin{array}{c|c} -J\dot{\theta}(t) & R(-\theta(t))\,\dot{d}(t) \\ \hline 0 & 0 \end{array}\right]$$

implicitly is $\dot{d}_B^A(t)$

$$\Rightarrow \dot{g}_B^B(t) = \left[\begin{array}{c|c} \hat{\dot{\theta}}(t) & R^{-1}(\theta(t))\,\dot{d}(t) \\ \hline 0 & 0 \end{array}\right] \quad \text{where } \hat{a} = -aJ$$

I can think of this as:

$$\underbrace{\dot{g}_B^B(t) = \left[\begin{array}{c|c} \hat{\dot{\theta}}(t) & \dot{d}_B^B(t) \\ \hline 0 & 0 \end{array}\right]}_{\text{called body velocity}}$$

Because $\dot{g}_B^B(t)$ depends on three variables only, it is often convenient to write it that way. Furthermore, there is another symbol for velocities of $SE(2)$ elements. It is $\xi$.

The velocity $\xi$ is written as:

$$\xi = \begin{Bmatrix} v \\ \omega \end{Bmatrix} = \begin{Bmatrix} v_x \\ v_y \\ \omega \end{Bmatrix}$$

vector form (corresponds to vector form of $g \in SE(2)$)

$$g = \begin{Bmatrix} x \\ y \\ \theta \end{Bmatrix}$$

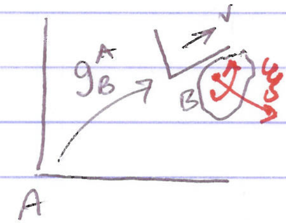
Ok, but what is $\xi$ in homogenous representation?

$$\hat{\xi} = \begin{bmatrix} \hat{\xi}_3 & \begin{Bmatrix} \xi_1 \\ \xi_2 \end{Bmatrix} \\ 0 & 0 \end{bmatrix} \qquad \xi = \begin{Bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{Bmatrix}$$

or $\hat{\xi} = \left[\begin{array}{c|c} \hat{\omega} & v \\ \hline 0 & 0 \end{array}\right]$ where $\xi = \begin{Bmatrix} v \\ \omega \end{Bmatrix}$

from homogenous form of vector to vector form of vector is "unhatting" and is written $\cdot^{\vee}$

RECAP:




if I see $V^A$, what is $V^B$

$$V^B = g_A^B \, V^A$$

$V = \begin{Bmatrix} V_x \\ V_y \\ 0 \end{Bmatrix}$ in homogenous form of 2D vector.

if I see $\xi^A$, what is $\xi^B$

$$\xi^B = g_A^B \cdot \xi^A$$ (in homogenous coords.

$$\hat{\xi}^B = g_A^B \cdot \hat{\xi}^A$$

Back to Properties:

1) Linearity

$$\xi_1 + \xi_2 = \begin{Bmatrix} v_1 \\ \omega_1 \end{Bmatrix} + \begin{Bmatrix} v_2 \\ \omega_2 \end{Bmatrix} = \begin{Bmatrix} v_1 + v_2 \\ \omega_1 + \omega_2 \end{Bmatrix}$$

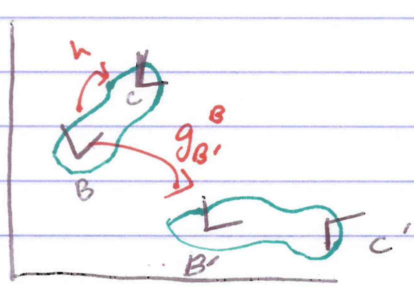
$$\hat{\xi}_1 + \hat{\xi}_2 = \begin{bmatrix} \hat{\omega}_1 & v \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} \hat{\omega}_2 & v \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \hat{\omega}_1 + \hat{\omega}_2 & v_1 + v_2 \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} (\omega_1 + \omega_2)^\wedge & (v_1 + v_2) \\ 0 & 0 \end{bmatrix}$$

$$\hat{\omega}_1 + \hat{\omega}_2 = -\omega_1 J^T + (-\omega_2) J^T$$
$$= -(\omega_1 + \omega_2) J^T$$
$$= (\omega_1 + \omega_2)^\wedge$$

$$= (\xi_1 + \xi_2)^\wedge$$

2) How does it transform under change of reference?




... To be continued