Exponential Representation of Group Motions / Displacements

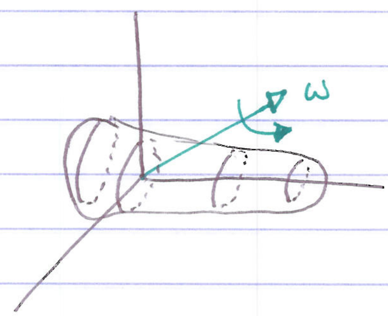
This is the last part of this section & covers the third important characteristic regarding group element and vectors.

Outline:

1) $SO(3)$ and $\mathfrak{so}(3)$
2) $SE(3)$ and $\mathfrak{se}(3)$

} Covers: exp, ln & their properties.

I. $SO(3)$ and $\mathfrak{so}(3)$



Consider a body undergoing rotation about the vector ω. In doing so, let's see what happens to a point on this body:

$$(1) \qquad \dot{q}(t) = \omega \times q(t) = \hat{\omega}\, q(t)$$

axis of rotation

cross-product is linear operation.

what is $\hat{\omega}$?

$$\begin{vmatrix} \omega_1 & q_1 & i \\ \omega_2 & q_2 & j \\ \omega_3 & q_3 & k \end{vmatrix} = (\omega_2 q_3 - \omega_3 q_2)i + (\omega_3 q_1 - \omega_1 q_3)j + (\omega_1 q_2 - \omega_2 q_1)k$$

$$= \begin{Bmatrix} \omega_2 q_3 - \omega_3 q_2 \\ \omega_3 q_1 - \omega_1 q_3 \\ \omega_1 q_2 - \omega_2 q_1 \end{Bmatrix} = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix}$$

Integrating equation (1):

$$q(t) = e^{\hat{\omega} t} q(0)$$

what is this?

The exponent of $\hat{\omega}$ for time τ is:

$$e^{\hat{\omega}\tau} = \mathbb{1} + \frac{\hat{\omega}}{\|\omega\|} \sin(\|\omega\|\tau) + \frac{\hat{\omega}^2}{\|\omega\|^2}\left(1 - \cos(\|\omega\|\tau)\right)$$

similar to the R-matrix

Rodrigues' Formula.

How do we know that $e^{\hat{\omega}\tau}$ is a rotation?
(e.g. is $e^{\hat{\omega}\tau} \in SO(3)$?)

if $R \in SO(3)$, then
a) $\det(R) = 1$
b) $R^T R = \mathbb{1}$

we need to show this for $e^{\hat{\omega}\tau}$

$Tr(A) = \sum_1^n A_{ii}$

a) $\det(e^{\hat{\omega}\tau}) = e^{Tr(\hat{\omega}\tau)} = e^0 = 1$

b) $(e^{\hat{\omega}\tau})^T (e^{\hat{\omega}\tau}) = e^{\hat{\omega}^T \tau} e^{\hat{\omega}\tau} = e^{-\hat{\omega}\tau} e^{\omega\tau} = e^{(\hat{\omega}\tau - \hat{\omega}\tau)}$
$= \mathbb{1}$

What about the other way around? Yes, but must be careful.
Proposition, the exponential map is surjective onto $SO(3)$. Given $R \in SO(3)$, there exists an $\omega \in \mathbb{R}^3$, where $\|\omega\| = 1$, and $\tau \in \mathbb{R}$ such that $R = \exp(\hat{\omega}\tau)$

Surjective : many to one ; onto : completely covers.

. Note that the proposition is like saying that there exists a function taking in a rotation R and returning a pair (ω, τ) such that

$$R = e^{\omega \tau} \quad \text{where} \quad \|\omega\| = 1$$

This function will be called the <u>logarith</u>, and will be denoted by $\ln$,

$$(\omega, \tau) = \ln R \quad \text{where} \quad \|\omega\| = 1$$

If we allow for $\|\omega\| = \tau$, then we will write

$$\omega = \ln R$$

The pair (ω, τ) are not necessarily unique!

Proof.

Proof is constructive

$$R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} \longleftarrow$$ only 3 of these entries are uniquely determined.

(ω, τ) also has 3 DOF!

So, let's see if we can solve the equation:

$$R = e^{\hat{\omega}\tau} = \mathbb{1} + \hat{\omega}\sin(\tau) + \hat{\omega}^2(1 - \cos(\tau))$$

(note: $\|\omega\| = 1$)

$\Rightarrow$ let, $S_\tau = \sin(\tau)$, $C_\tau = \cos(\tau)$, and $v_\tau = (1 - \cos(\tau))$

$$e^{\hat{\omega}\tau} = \cancel{1(-v_\tau((\omega_2)^2 + (\omega_3)^2)} \longrightarrow$$

$$e^{\hat{\omega}\tau} = \begin{bmatrix} 1 - v_\tau((\omega_2)^2 + (\omega_3)^2) & \omega_1\omega_2 \tau - \omega_3 s_\tau & \omega_1\omega_3 v_\tau + \omega_2 s_\tau \\ \omega_1\omega_2 v_\tau + \omega_3 s_\tau & 1 - v_\tau((\omega_1)^2 + (\omega_3)^2) & \omega_2\omega_3 v_\tau - \omega_1 s_\tau \\ \omega_1\omega_3 v_\tau - \omega_2 s_\tau & \omega_2\omega_3 v_\tau + \omega_1 s_\tau & 1 - v_\tau((\omega_1)^2 + (\omega_2)^2) \end{bmatrix}$$

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Continued from last time:

$$e^{\hat{\omega}\tau} = \begin{bmatrix} (\omega_1)^2 v_\tau + c_\tau & \omega_1\omega_2 v_\tau - \omega_3 s_\tau & \omega_1\omega_3 v_\tau + \omega_2 s_\tau \\ \omega_1\omega_2 v_\tau + \omega_3 s_\tau & (\omega_2)^2 v_\tau + c_\tau & \omega_2\omega_3 v_\tau - \omega_1 s_\tau \\ \omega_1\omega_3 v_\tau - \omega_2 s_\tau & \omega_2\omega_3 v_\tau + \omega_1 s_\tau & (\omega_3)^2 v_\tau + c_\tau \end{bmatrix}$$

where $c_\tau = \cos\tau$,
$s_\tau = \sin\tau$
$v_\tau = 1 - \cos\tau$

↑ can see symmetric and skew-symmetric parts.

$(\Lambda^T = \Lambda)$ $(\Sigma^T = -\Sigma)$

Equating the elements of $e^{\hat{\omega}\tau}$ to elements of R should give answer to what ω & τ need to be.

Step 1] examine the trace of the two matrices:

$$Tr(R) = r_{11} + r_{22} + r_{33}$$

$$Tr(e^{\hat{\omega}\tau}) = (\omega_1^2 + \omega_2^2 + \omega_3^2) v_\tau + 3c_\tau = v_\tau + 3c_\tau.$$
$$= 1 + 2\cos\tau$$

$$\Rightarrow 1 + 2\cos(\tau) = Tr(R)$$

$$\Rightarrow \cos(\tau) = \frac{Tr(R) - 1}{2}$$

$$\Rightarrow \tau = \cos^{-1}\left(\frac{Tr(R) - 1}{2}\right)$$

range of solution is $[0, \pi]$, but in reality adding $2\pi k$ can give a valid answer too.

if $\tau = 0$, then ω can be anything
if $\tau \neq 0$, the need to find ω.

Step 2) So, I mentioned that $e^{\hat{\omega}\tau}$ can be broken up into two special matrices

$$e^{\hat{\omega}\tau} = \Lambda + \Sigma \quad \text{where } \Lambda^T = \Lambda \;\&\; \Sigma^T = -\Sigma$$

Therefore, let's examine $R - R^T$ vs. $e^{\hat{\omega}\tau} - (e^{\hat{\omega}\tau})^T$

$$e^{\hat{\omega}\tau} - (e^{\hat{\omega}\tau}) = (\Lambda + \Sigma) - (\Lambda - \Sigma)^T = \Lambda + \Sigma - \Lambda^T - \Sigma^T$$
$$= 2\Sigma$$
$$= \begin{bmatrix} 0 & -2\omega_3 S\tau & 2\omega_2 S\tau \\ 2\omega_3 S\tau & 0 & -2\omega_1 S\tau \\ -2\omega_2 S\tau & 2\omega_1 S\tau & 0 \end{bmatrix} = 2\sin(\tau)\hat{\omega}$$

Equating the unique terms gives

$$r_{32} - r_{23} = 2\omega_1 \sin\tau$$

$$r_{13} - r_{31} = 2\omega_2 \sin\tau$$

$$r_{21} - r_{12} = 2\omega_3 \sin\tau$$

$$\Rightarrow \omega = \frac{1}{2\sin(\tau)} \begin{Bmatrix} r_{32} - r_{23} \\ r_{13} - r_{31} \\ r_{21} - r_{12} \end{Bmatrix}$$

In summary:

Exponential: Given $\omega \in \mathbb{R}^3$ such that $\hat{\omega} \in so(3)$, then

$$e^{\hat{\omega}\tau} = \mathbb{1} + \frac{\hat{\omega}}{\|\omega\|}\sin(\|\omega\|\tau) + \frac{\hat{\omega}^2}{\|\omega\|^2}\left(1 - \cos(\|\omega\|\tau)\right) \in SO(3)$$

where $\hat{\omega} = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}$

Given $R \in SO(3)$, $\exists\, \omega \in \mathbb{R}^3$, $\tau \in \mathbb{R}$ where $\|\omega\| = 1$ defined by

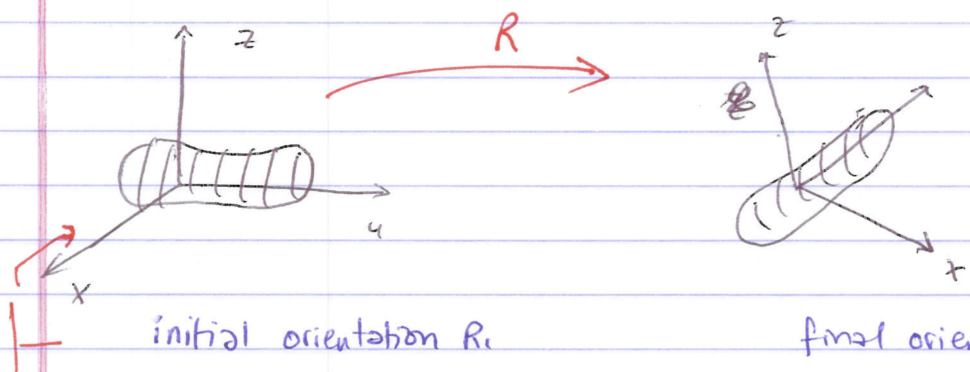
$$\tau = \cos^{-1}\left(\frac{\text{trace}(R) - 1}{2}\right)$$

if $\tau = 0$, pick some ω (can be anything)

if $\tau \neq 0$, $\omega = \dfrac{1}{2\sin(\tau)} \begin{Bmatrix} r_{32} - r_{23} \\ r_{13} - r_{31} \\ r_{21} - r_{12} \end{Bmatrix}$

which ~~can be~~ when written as a function is called the Logarithm,

$$(\omega, \tau) = \ln R$$



$R_2 = R_1 R$

initial orientation R_1

final orientation R_2

Change in orientation is $R = R_1^{-1} R_2$

$R_1^{-1} R_2$

$\Downarrow$

under logarithm get converted into a vector such that $R_2 = R_1 e^{\hat{\omega}\tau}$

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II. Exponential & Logarithm of SE(3) / $\hat{SE}(3)$

For this case $\xi \in \mathbb{R}^6$

$$\xi = \begin{Bmatrix} v \\ \omega \end{Bmatrix} \quad \text{and} \quad \hat{\xi} = \left[\begin{array}{c|c} \hat{\omega} & v \\ \hline 0 & 0 \end{array}\right]$$

$$v \in \mathbb{R}^3, \ \omega \in \mathbb{R}^3$$

The exponential of ξ is:

$$e^{\hat{\xi}\tau} = \begin{cases} \text{if } \omega = 0, \text{ then} & e^{\hat{\xi}\tau} = \left[\begin{array}{c|c} \mathbb{1} & v\tau \\ \hline 0 & 1 \end{array}\right] \\ \text{otherwise} & e^{\hat{\xi}\tau} = \left[\begin{array}{c|c} e^{\hat{\omega}\tau} & (\mathbb{1} - e^{\hat{\omega}\tau})\frac{\hat{\omega}}{\|\omega\|^2} v + \frac{\omega\omega^T}{\|\omega\|^2} v\tau \\ \hline & 1 \end{array}\right] \end{cases}$$

re-written

$$e^{\hat{\xi}\tau} = \left[\begin{array}{c|c} e^{\hat{\omega}\tau} & (\mathbb{1} - e^{\hat{\omega}\tau})\frac{\hat{\omega}}{\|\omega\|^2} v + \frac{\omega\omega^T}{\|\omega\|^2} v\tau \\ \hline 0 & 1 \end{array}\right]$$

Alternative form for the translational component associated to the exponential:

$$(\mathbb{1} - e^{\hat{\omega}\tau})\frac{\hat{\omega}}{\|\omega\|^2} v + \frac{\omega\omega^T}{\|\omega\|^2} v\tau$$

$$= \left[\mathbb{1} - \left(\mathbb{1} + \frac{\hat{\omega}}{\|\omega\|}\sin(\|\omega\|\tau) + \frac{\hat{\omega}^2}{\|\omega\|^2}(1 - \cos(\|\omega\|\tau))\right)\right]\frac{\hat{\omega}}{\|\omega\|^2} v + \frac{\omega\omega^T v\tau}{\|\omega\|^2}$$

$$= -\left(\frac{\hat{\omega}^2}{\|\omega\|^3}\sin(\|\omega\|\tau) + \frac{\hat{\omega}^3}{\|\omega\|^4}(1 - \cos(\|\omega\|\tau))\right) v + \frac{\omega}{\|\omega\|}\frac{\omega^T}{\|\omega\|} v\tau$$

$$= \cancel{\frac{\|\omega\|^2 \hat{\omega}}{\|\omega\|^4}\left(1-\cos(\|\omega\|\tau)\right)v - \frac{\hat{\omega}^2}{\|\omega\|^3}\left(\|\omega\|\tau - \sin(\|\omega\|\tau)\right)v}$$

How did I get here?

$\hat{\omega}^3 = -\|\omega\|^2 \hat{\omega}$

$\hat{\omega}^2 = \omega\omega^T - \|\omega\|^2 \mathbb{1}$

$J^1 \cdot J = -\mathbb{1}$

$J^{1^2} \cdot J = -J^1$

$J^{1^4} = \mathbb{1}$

$J^{1^5} = J^1$

$$= \frac{\cancel{\|\omega\|^2}}{\|\omega\|^{\cancel{4}2}} \hat{\omega}\left(1-\cos(\|\omega\|\tau)\right)v - \frac{\hat{\omega}^2}{\|\omega\|^3}\sin(\|\omega\|\tau)v + \left(\mathbb{1} + \frac{\hat{\omega}^2}{\|\omega\|^2}\right)v\tau$$

$$= \mathbb{1}v\tau + \frac{\hat{\omega}}{\|\omega\|^2}\left(1-\cos(\|\omega\|\tau)\right)v + \frac{\hat{\omega}^2}{\|\omega\|^3}\left(\|\omega\|\tau - \sin(\|\omega\|\tau)\right)v$$

$$= \left[\mathbb{1}\tau + \frac{\hat{\omega}}{\|\omega\|^2}\left(1-\cos(\|\omega\|\tau)\right) + \frac{\hat{\omega}^2}{\|\omega\|^3}\left(\|\omega\|\tau - \sin(\|\omega\|\tau)\right)\right]v$$

Theorem (Chasles): Every rigid body motion can be realized by a rotation about an axis combined with a translation parallel to the axis.

→ proof is ~~the~~ the Logarithm.

The Logarithm.

Given $g = \begin{bmatrix} R & d \\ 0 & 1 \end{bmatrix} \in SE(3)$, what is $(\xi, \tau) = \ln g$?

Recall that $\xi = \begin{Bmatrix} v \\ \omega \end{Bmatrix}$, so we need to figure out v & ω

1. $(\omega, \tau) = \ln R$

2. Solve for v. (given $\omega \neq 0$)
we just compare translation of g to translation of $e^{\hat{\xi}\tau}$.

$$d = (\mathbb{1} - R)\frac{\hat{\omega}}{\|\omega\|^2} v + \frac{\omega\omega^T}{\|\omega\|^2} v\tau$$

$$d = \left[(\mathbb{1} - R)\frac{\hat{\omega}}{\|\omega\|^2} + \frac{\omega\omega^T}{\|\omega\|^2}\tau \right] v$$

$$\Rightarrow v = \left[(\mathbb{1} - R)\frac{\hat{\omega}}{\|\omega\|^2} + \frac{\omega\omega^T}{\|\omega\|^2}\tau \right]^{-1} \cdot d$$

$$= \|\omega\|^2 \left[(\mathbb{1} - R)\hat{\omega} + \omega\omega^T\tau \right]^{-1} d$$

3. if $\omega = 0$, then

$v = d$, $\tau = 1$ OR $v = \frac{d}{\|d\|}$, $\tau = \|d\|$

unit time vs. unit velocity