ECE 4560

Manipulator Jacobian continued ...

let $\alpha = \alpha(t)$, then end-effector trajectory is $g_e(\alpha(t))$
$\Rightarrow$ take time-derivative:

$$\frac{d}{dt}\left[g_e(\alpha(t))\right] = \frac{\partial g_e}{\partial \alpha} \cdot \frac{\partial \alpha}{\partial t} = Dg_e(\alpha(t)) \cdot \dot{\alpha}(t)$$

where $\dot{\alpha}(t) = \frac{d\alpha}{dt}$

$$\frac{\partial g_e}{\partial \alpha} = \begin{bmatrix} \partial g_e^1/\partial\alpha^1 & \cdots & \partial g_e^1/\partial\alpha^m \\ \vdots & & \\ \partial g_e^n/\partial\alpha^1 & \cdots & \partial g_e^n/\partial\alpha^m \end{bmatrix}$$

for $\alpha = (\alpha^1, \ldots, \alpha^m)^T$
$g_e = (g_e^1, \ldots, g_e^n)^T$

$$\dot{g}_e = Dg_e \cdot \dot{\alpha}$$

. but, what do the partial derivatives mean when the representation for g_e is in homogeneous coordinates (AKA homogeneous vector form)?

the first thing to do is switch frames, because $\dot{g}_e = Dg_e \cdot \dot{\alpha}$ has mixed frames.

body frame: $J^b(\alpha) = g_e^{-1}(\alpha) \cdot \frac{\partial g_e(\alpha)}{\partial \alpha}$ body manipulator Jacobian

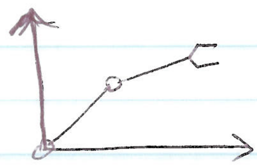
spatial frame: $J^s(\alpha) = \frac{\partial g_e(\alpha)}{\partial \alpha} \cdot g_e^{-1}(\alpha)$ spatial manipulator Jacobian.

be careful about $\cdot$ (dot)

in mixed frames, $J(\alpha) = \frac{\partial g_e(\alpha)}{\partial \alpha}$ is called the manipulator Jacobian.

. in practice, we work in homogeneous form where needed or when the computations are simpler, then convert to vector form by unhatting the result.

Example



$$g_e(\alpha) = g_1(\alpha_1) \cdot g_2(\alpha_2)\, g_3$$

$$= \begin{bmatrix} \cos(\alpha_1) & -\sin(\alpha_1) & 0 \\ \sin(\alpha_1) & \cos(\alpha_1) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos(\alpha_2) & -\sin(\alpha_2) & l_1 \\ \sin(\alpha_2) & \cos(\alpha_2) & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & l_2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$J^b(\alpha) = g_e^{-1}(\alpha) \cdot \frac{D g_e(\alpha)}{\cancel{g}} = g_e^{-1}(\alpha) \cdot \frac{\partial g_e}{\partial \alpha}$$

$$= g_e^{-1}(\alpha) \cdot \begin{bmatrix} \frac{\partial g_e}{\partial \alpha^1} & \frac{\partial g_e}{\partial \alpha^2} \end{bmatrix}$$

$$= g_e^{-1}(\alpha) \begin{bmatrix} \frac{\partial g_1}{\partial \alpha_1} g_2 g_3 & g_1 \frac{\partial g_2}{\partial \alpha_2} g_3 \end{bmatrix}$$

$$= g_3^{-1} g_2^{-1}(\alpha_2)\, g_1^{-1}(\alpha_1) \begin{bmatrix} \frac{\partial g_1}{\partial \alpha_1} g_2 g_3 & g_1 \frac{\partial g_2}{\partial \alpha_2} g_3 \end{bmatrix}$$

$$= \begin{bmatrix} g_3^{-1} g_2^{-1} g_1^{-1} \frac{\partial g_1}{\partial \alpha_1} g_2 g_3 & g_3^{-1} g_2^{-1} \cancel{g_1^{-1} g_1} \cdot \frac{\partial g_2}{\partial \alpha_2} g_3 \end{bmatrix}$$

$$= \begin{bmatrix} g_3^{-1} g_2^{-1} \underbrace{\left(g_1^{-1} \frac{\partial g_1}{\partial \alpha_1} \right)}_{\text{body term}} g_2 g_3 & g_3^{-1} \underbrace{\left(g_2^{-1} \frac{\partial g_2}{\partial \alpha_2} \right)}_{\text{body term}} g_3 \end{bmatrix}$$

body term

$$\begin{bmatrix} \cos(\alpha_1) & +\sin(\alpha_1) & 0 \\ -\sin(\alpha_1) & \cos(\alpha_1) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -\sin(\alpha_1) & \cos(\alpha_1) & 0 \\ -\cos(\alpha_1) & \sin(\alpha_1) & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ +1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

this term also comes out to:

$$\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\vee$ = unhatting

$$= \left[g_3^{-1} g_2^{-1} \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} g_2 g_3 \qquad g_3^{-1} \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} g_3 \right]$$

$$= \left[Ad_{g_3^{-1}} Ad_{g_2^{-1}} \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}^{\vee} \qquad Ad_{g_3^{-1}} \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}^{\vee} \right]$$

$$= \left[Ad_{g_3^{-1} g_2^{-1}} \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}^{\vee} \qquad Ad_{g_3^{-1}} \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}^{\vee} \right]$$

$$= \left[Ad_{(g_2 g_3)^{-1}} \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}^{\vee} \qquad Ad_{g_3^{-1}} \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}^{\vee} \right]$$

$$J^b(\alpha) = \left[Ad_{(g_2 g_3)^{-1}} \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix} \qquad Ad_{g_3^{-1}} \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix} \right]$$

IMPORTANT NOTE: ORDER OF IMPLEMENTATION IS IRRELEVANT,

$(Ad_h \xi)^{\vee} = Ad_h (\hat{\xi}^{\vee})$

$(h \hat{\xi} h^{-1})^{\vee}$ vs. $\underbrace{Ad_h}$ ξ

can be written as a special matrix.

So, now to get body velocity of end-effector as a function of joint velocities ($\dot{\alpha}$),

$$\xi^b = J^b(\alpha) \cdot \dot{\alpha}$$

$$\xi^b = Ad_{(g_2 g_3)^{-1}} \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix} \cdot \dot{\alpha}_1 + Ad_{g_3^{-1}} \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix} \cdot \dot{\alpha}_2$$

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1. Product of Lie Groups

Consider $g_e(\alpha) = g_1(\alpha_1) \cdots g_m(\alpha_m) g_{m+1}$
and let joint configuration be a function of time,
$\alpha = \alpha(t)$.
Take time derivative.

$$\frac{d}{dt} g_e(\alpha(t)) = \left(\frac{d}{dt} g_1(\alpha_1(t))\right) g_2(\alpha_2(t)) \cdots g_m(\alpha_m(t)) + \cdots$$

$$g_1(\alpha_1(t)) \frac{d}{dt} g_2(\alpha_2(t)) \cdots g_m(\alpha_m(t)) g_{m+1} + \cdots + g_1(\alpha_1(t)) \cdots g_{m-1}(\alpha_{m-1}(t)) \frac{d}{dt} g_m(\alpha_m(t)) g_{m+1}$$

We have that $\frac{d}{dt} g_i(\alpha_i(t)) = \frac{\partial g_i}{\partial \alpha_i} \cdot \dot{\alpha}_i(t)$

so, for simplicity define $J_i(\alpha_i) = \frac{\partial g_i}{\partial \alpha_i}$

$\Rightarrow \frac{d}{dt} g_i(\alpha_i(t)) = J_i(\alpha_i(t)) \cdot \dot{\alpha}_i(t)$

This definition leads to

$$\frac{d}{dt} g_e(\alpha(t)) = J_1(\alpha_1(t))\dot{\alpha}_1(t) \prod_{j=2}^{m+1} g_j(\alpha_j(t)) + g_1(\alpha_1(t)) J_2(\alpha_2(t)) \dot{\alpha}_2(t) \prod_{j=3}^{m+1} g_j(\alpha_j(t)) + \cdots + \left(\prod_{j=1}^{k-1} g_j(\alpha_j(t))\right) J_k(\alpha_k(t)) \dot{\alpha}_k(t) \left(\prod_{j=k+1}^{m+1} g_j(\alpha_j(t))\right) + \cdots + \left(\prod_{j=1}^{m-1} g_j(\alpha_j(t))\right) J_m(\alpha_m(t)) \dot{\alpha}_m(t) g_{m+1}$$

Let's move to the body frame

$$\hat{\xi}_e^b(\alpha(t)) = g_e^{-1}(\alpha(t))\frac{d}{dt} g_e(\alpha(t))$$

now, the forward kinematics are $g_e = \prod_{j=1}^{m+1} g_j$

and the inverse ~~kinematics~~ of g_e is $g_e^{-1} = \prod_{j=m+1}^{1} g_j^{-1}$

$$\hat{\xi}_e^b(\alpha(t)) = \left(\prod_{j=m+1}^{1} g_j^{-1}\right) J_1(\alpha_1)\dot{\alpha}_1 \left(\prod_{j=2}^{m+1} g_j\right) +$$

$$\left(\prod_{j=m+1}^{1} g_j^{-1}\right) g_1 J_2(\alpha_2)\dot{\alpha}_2 \left(\prod_{j=3}^{m+1} g_j\right) +$$

cancellation

$$\cdots + \left(\prod_{j=m+1}^{1} g_j^{-1}\right)\left(\prod_{j=1}^{k-1} g_j\right) J_k(\alpha_k)\dot{\alpha}_k \left(\prod_{j=k+1}^{m+1} g_j\right) +$$

cancellation

$$\cdots + \left(\prod_{j=m+1}^{1} g_j^{-1}\right)\left(\prod_{j=1}^{m-1} g_j\right) J_m(\alpha_m)\dot{\alpha}_m g_{m+1}$$

cancellation.

$$= \left(\prod_{j=m+1}^{2} g_j^{-1}\right) g_1^{-1} J_1(\alpha_1)\dot{\alpha}_1 \left(\prod_{j=2}^{m+1} g_j\right) + \left(\prod_{j=m+1}^{3} g_j^{-1}\right) g_2^{-1} J_2(\alpha_2)\dot{\alpha}_2 \left(\prod_{j=2}^{m+1} g_j\right) + \cdots + \left(\prod_{j=m+1}^{k+1} g_j\right) g_k^{-1} J_k(\alpha_k)\dot{\alpha}_k \left(\prod_{j=k+1}^{m+1} g_j\right) +$$

$$\cdots + g_{m+1}^{-1}\, g_m^{-1} J_m(\alpha_m)\dot{\alpha}_m\, g_{m+1}$$

– ~~black~~ red : body form

– ~~green~~ green: adjoint (inverse)

Now define $J_i^b(\alpha_i) \equiv g_i^{-1}(\alpha_i) J_i(\alpha_i) = g_i^{-1}(\alpha_i) \dfrac{\partial g_i(\alpha_i)}{\partial \alpha_i}$

body form of $J_i(\alpha_i)$

What does it mean? Well $\dfrac{d}{dt} g_i(\alpha_i(t)) = J_i(\alpha_i)\dot{\alpha}_i$

$\Rightarrow$ into joint's body frame.

$$\xi_i^b(\alpha_i(t)) = J_i^b(\alpha_i)\dot{\alpha}_i$$

local effect of adjusting joint i. have simple form.

$$\Rightarrow \hat{\xi}_e^b(\alpha(t)) = \mathrm{Ad}^{-1}_{\left(\prod_{j=2}^{m+1} g_j\right)} J_1^b(\alpha_1)\dot{\alpha}_1 + \dots + \mathrm{Ad}^{-1}_{\left(\prod_{j=k+1}^{m+1} g_j\right)} J_k^b(\alpha_k)\dot{\alpha}_k + \dots + \mathrm{Ad}^{-1}_{g_m} J_m^b(\alpha_m)\dot{\alpha}_m$$

$\Rightarrow$ factor

$$\hat{\xi}_e^b(\alpha(t)) = \underbrace{\left[\mathrm{Ad}^{-1}_{\left(\prod_{j=2}^{m+1} g_j\right)} J_1^b(\alpha_1), \dots, \mathrm{Ad}^{-1}_{\left(\prod_{j=k+1}^{m+1}\right)} J_k^b(\alpha_k), \dots, \mathrm{Ad}^{-1}_{g_m} J_m^b(\alpha_m)\right]}_{\text{each of these should be column vectors.}} \dot{\alpha}$$

$$\hat{\xi}_e^b(\alpha(t)) = J_e^b(\alpha(t))\dot{\alpha}(t)$$

to get spatial Jacobian from body Jacobian,
apply adjoint of $g_e(\alpha)$

$$J^s(\alpha) = Ad_{g_e} J^b(\alpha)$$

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body Jacobian from Wednesday:

$$J^b(\alpha) = \left[Ad^{-1}_{\left(\prod_{j=2}^{m+1} g_j\right)} J_1^b \;\cdots\; Ad^{-1}_{\left(\prod_{j=k+1}^{m+1}\right)} J_k^b \;\cdots\; Ad^{-1}_{g_{m+1}} J_m^b \right]$$

to move to spatial, take adjoint of body.

Now look at J^b, we see that taking adjoint will involve terms of the form:

$$Ad_{g_e(\alpha)} Ad^{-1}_{\left(\prod_{j=k+1}^{m+1} g_j\right)} = Ad_{\left(\prod_{j=1}^{m+1} g_j\right)} Ad^{-1}_{\left(\prod_{j=k+1}^{m+1} g_j\right)}$$

$$= Ad_{\left(\prod_{j=1}^{m+1} g_j\right)} Ad_{\left(\prod_{j=m+1}^{k+1} g_j^{-1}\right)}$$

cancellation will occur.

$$= Ad_{\left(\prod_{j=1}^{k} g_j\right)}$$

$\Rightarrow$ plug it in

$$(*) \quad J^s(\alpha) = Ad_{g_e(\alpha)} J^b(\alpha) = \left[Ad_{g_1} J_1^b \;\cdots\; Ad_{\left(\prod_{j=1}^{k} g_j\right)} J_k^b \;\cdots\; Ad_{\left(\prod_{j=1}^{m+1} g_j\right)} J_m^b \right]$$

Recall: $J_i^b = g_i^{-1}(\alpha_i) J_i(\alpha_i)$

$\Rightarrow J_i^s = Ad_{g_i} J_i^b = J_i g_i^{-1}(\alpha_i)$

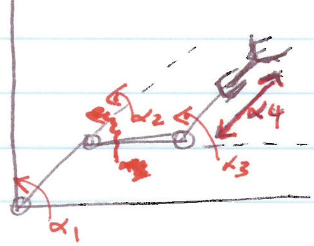
Now, apply this to $(*)$

$$\Rightarrow \quad J^s(\alpha) = \begin{bmatrix} | & & | & & | \\ Ad_{g_1} J_1^b & \cdots & Ad_{\left(\prod_{j=1}^{k-1} g_j\right)} Ad_{g_k} J_k^b & \cdots & Ad_{\left(\prod_{j=1}^{m-1} g_j\right)} Ad_{g_m} J_m^b \\ | & & | & & | \end{bmatrix}$$

$$J^s(\alpha) = \begin{bmatrix} | & & | & & | \\ J_1^s(\alpha_1) & \cdots & Ad_{\left(\prod_{j=1}^{k-1} g_j\right)} J_k^s(\alpha_k) & \cdots & Ad_{\left(\prod_{j=1}^{m-1} g_j\right)} J_m^s(\alpha_m) \\ | & & | & & | \end{bmatrix}$$

* product of Lie groups best suited for body Jacobian, but still fine for spatial.

Example



Will compute Jacobian using product of Lie groups.

$$g_e(\alpha) = g_1(\alpha_1)\, g_2(\alpha_2)\, g_3(\alpha_3)\, g_4(\alpha_4)\, g_5$$

g_5 normally is here, but since it's the identity, we'll ignore it.

- need the individual g_i's to find J_i^b's.

$$g_1(\alpha_1) = \begin{bmatrix} R(\alpha_1) & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow J_1^b(\alpha_1) = \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix}$$

*no translation in g

*there is a rotational component in $g_1(\alpha_1) \Rightarrow 1$

$$g_2(\alpha_2) = \begin{bmatrix} R(\alpha_2) & \begin{Bmatrix} l_1 \\ 0 \end{Bmatrix} \\ 0 & 1 \end{bmatrix} \Rightarrow J_2^b(\alpha_2) = \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix}$$

*since the translation component doesn't vary w/ α_2.

$$g_3(\alpha_3) = \begin{bmatrix} R(\alpha_3) & \begin{Bmatrix} l_2 \\ 0 \end{Bmatrix} \\ 0 & 1 \end{bmatrix} \Rightarrow J_3^b(\alpha_3) = \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix}$$

$\longrightarrow$

$$g_4(\alpha_4) = \left[\begin{array}{c|c} \mathbb{1} & \begin{Bmatrix}\alpha_4\\0\end{Bmatrix} \\ \hline 0 & 1\end{array}\right] \Rightarrow J_4^b = \begin{Bmatrix}1\\0\\ \hline 0\end{Bmatrix}$$

$\Rightarrow$ to work it out :

$$J_2^b(\alpha_2) = g_2^{-1}(\alpha_2)\frac{\partial g}{\partial \alpha_2} = \left(\left[\begin{array}{c|c} R^T(\alpha_2) & -R^T(\alpha_2)d_2 \\ \hline 0 & 1\end{array}\right]\left[\begin{array}{c|c} DR(\alpha_2) & 0 \\ \hline 0 & 0\end{array}\right]\right)^\vee$$

$$= \left[\begin{array}{c|c} R^T(\alpha_2)\,DR(\alpha_2) & 0 \\ \hline 0 & 0\end{array}\right]^\vee$$

$$= \left[\begin{array}{c|c} J' & 0 \\ \hline 0 & 0\end{array}\right]^\vee = \begin{Bmatrix}0\\0\\ \hline 1\end{Bmatrix}$$