Wrenches

a generalized force acting on a rigid body consists of

a linear component (pure force), and

an angular component (pure moment)

acting at a point.

this is important.

This generalized force is called a <u>wrench</u> and is a vector in $\mathbb{R}^6$ (for $SE(3)$):

$$F = \begin{Bmatrix} f \\ \tau \end{Bmatrix} \qquad \begin{array}{ll} f \in \mathbb{R}^3 & \text{linear part} \\ \tau \in \mathbb{R}^3 & \text{angular part} \end{array}$$

- a force/moment pair.
- values will depend on the coordinate frame.

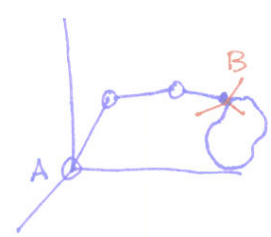
Properties of wrenches:

1) combine w/ twists to define <u>instantaneous work</u>,

$$\delta W = F_B \, \xi^B = (f \cdot v + \tau \cdot \omega)$$

instantaneous body velocity (frame B)

applied wrench in frame B. at point B.



- both operate at the same point.

2) over time (1) generates work: $\quad W = \int_{t_1}^{t_2} F_B \cdot \xi^B \, dt$

3) two wrenches are ~~equit~~ equivalent if they do the same work independent of ~~ξ_x~~ ξ (wrenches need not be at same point).

can use to go from body to spatial

$$F_A = Ad^T_{g_{BA}} F_B$$

$$F^{spatial} = Ad^T_{g^{-1}} F^{body}$$

$$= (Ad^T_g)^{-1} F^{body}$$

$$F^{body} = Ad^T_g F^{spatial}$$

this is flipped compared to what we are used to.

- recall, that this is as though new frame were rigidly attached to old frame. This is NOT the same as just changing frame of reference of component description.

- wrenches can be added, but only if applied at the same point

Theorem (Poinsot). Every collection of wrenches applied to a rigid body is equivalent to a force applied ~~to~~ along a fixed axis plus a torque about the same axis

- dual theorem to Chasles' theorem.

$\xi = g^{-1}\dot{g}$

$$\boxed{\cancel{\xi^{body} = Ad_g \xi^{spatial}}}$$

$$\begin{bmatrix} R^T & -R^T p \\ 0 & 1 \end{bmatrix} \dot{g} =$$

R

$-R(\cdot)^\wedge$

$$\xi^{spatial} = Ad_g \xi^{body}$$

$$F^{sp} \cdot \xi^{sp} = F^b \xi^b$$

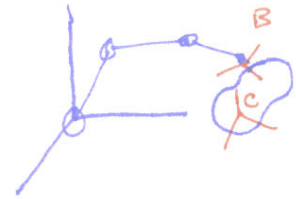
$$F^{sp} \cdot Ad_g \xi^b = F^b \xi^b$$

$$Ad_g^T F^{sp} = F^{body}$$

$\Rightarrow$ F_B^A - wrench ~~in B's frame~~ (at point B)

F_C^A - wrench ~~in C's frame~~ (at point C)

both in A frame



ξ_B^A - twist at point B

ξ_C^A - twist at point C

equivalency means:

$$F_C \cdot \xi_C = F_B \cdot \xi_B$$

$$= F_B \cdot \left(Ad_{g_{BC}} \xi_C \right)$$

$$= \left(Ad_{g_{BC}}^T F_B \right) \cdot \xi_C$$

$\Rightarrow$

$$F_C = Ad_{g_{BC}}^T F_B$$

transformation of wrench under coordinate frame transformation.

$$\begin{Bmatrix} f_C \\ \tau_C \end{Bmatrix} = \begin{bmatrix} R_{BC}^T & 0 \\ -R_{BC}^T \hat{p}_{BC} & R_{BC}^T \end{bmatrix} \begin{Bmatrix} f_B \\ \tau_B \end{Bmatrix} \quad \text{for } g_{BC} = (p_{BC}, R_{BC}).$$

you've seen this before:

pure force at B $\longrightarrow$ induces torque at C

Manipulator Dynamics

now, we know how to express the kinetic energy of a rigid body.

we already know how to express the potential energy of a rigid body.

how can this translate to a manipulator?

- it is a collection of rigid bodies w/ constraints
- can be described by the joint space
- if possible to define a Lagrangian, equations of motion follow ~~next~~ naturally.

Kinetic Energy:

- just need to sum up kinetic energy of each ~~E~~ link.

let the center of mass of each link be $g_{ci}(\vec{\theta})$

$\Rightarrow$

body velocity of link center-of-mass is

$$\xi_{ci}^{body} = J_{ci}^{body}(\theta)\dot{\theta}$$

$\Rightarrow$

kinetic energy of link is

$$T_i = \frac{1}{2}(\xi_{ci}^{body})^T M_i (\xi_{ci}^{body})$$

gen. inertia matrix of ith link.

$\Rightarrow$

$$T_i(\theta,\dot{\theta}) = \frac{1}{2}\dot{\theta}^T (J_{ci}^{body}(\theta))^T M_i J_{ci}^{body}(\theta)\dot{\theta}$$

$\Rightarrow$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot\theta} - \frac{\partial L}{\partial \theta} = \sum_{j=1}^{n} M_{ij}(\theta)\,\ddot\theta_j + \sum_{j,k=1}^{n}\left(\frac{\partial M_{ij}}{\partial \theta_k}\,\dot\theta_j\dot\theta_k - \frac{1}{2}\frac{\partial M_{kj}}{\partial \theta_i}\,\dot\theta_k\dot\theta_j\right) + \frac{\partial V}{\partial \theta_i}(\theta) = \Upsilon_i$$

Typically written as:

$$\sum_{j=1}^{n} M_{ij}(\theta)\,\ddot\theta_j + \sum_{j,k=1}^{n} \Gamma_{ijk}(\theta)\,\dot\theta_j\dot\theta_k + \frac{\partial V}{\partial \theta_i}(\theta) = \Upsilon_i$$

where,

$$\Gamma_{ijk}(\theta) = \frac{1}{2}\left(\frac{\partial M_{ij}(\theta)}{\partial \theta_k} + \frac{\partial M_{ik}(\theta)}{\partial \theta_j} - \frac{\partial M_{kj}(\theta)}{\partial \theta_i}\right)$$

called Christoffel symbols corresponding to mass matrix $M(\theta)$.

due to symmetry of M, $M = M^T$.

$$M(\theta)\,\ddot\theta + C(\theta,\dot\theta)\,\dot\theta + N(\theta,\dot\theta) = \tau$$

inertial forces

Coriolis & centrifugal forces

potential & non-conservative forces

actuator torques

Υ_i broken up into two parts.

- second-order, vector-differential equation.

$$C(\theta,\dot\theta) = \sum_{k=1}^{n}\Gamma_{ijk}(\theta)\,\dot\theta_k = \frac{1}{2}\sum_{k=1}^{n}\left(\frac{\partial M_{ij}}{\partial \theta_k} + \frac{\partial M_{ik}}{\partial \theta_j} - \frac{\partial M_{kj}}{\partial \theta_i}\right)\dot\theta_k$$

$$N_i(\theta,\dot\theta) = \frac{\partial V}{\partial \theta_i} - \text{external forces}$$

obtained from Υ_i

e.g. $\beta_i\dot\theta_i$

friction at joint.

Total kinetic energy found by summing all of the link KE's:

$$T(\theta,\dot\theta) = \sum_{i=1}^{n} T_i(\theta,\dot\theta) \equiv \frac{1}{2}\dot\theta^T M \dot\theta$$

$M(\theta) \in \mathbb{R}^{n\times n}$ is called manipulator inertia matrix.

$$M(\theta) = \sum_{i=1}^{n} \left(J_{ci}^{body}(\theta)\right)^T M_i \, J_{ci}^{body}(\theta)$$

Potential Energy:

- this one is simply a matter of substituting in the forward kinematics

$$V_i(g) \longrightarrow V_i(g_{ci}(\theta))$$

$$\Rightarrow \quad V(\theta) = \sum_{1}^{n} V_i(g_{ci}(\theta)) = \sum_{1}^{n} V_i \circ g_{ci}(\theta)$$

~~Total Energy~~:

Complete Manipulator Lagrangian:

$$L(\theta,\dot\theta) = \sum_{i=1}^{n} \left(T_i(\theta,\dot\theta) - V_i(\theta)\right) = \frac{1}{2}\dot\theta^T M(\theta)\dot\theta - V(\theta)$$

Equations of Motion:

Once we have the manipulator Lagrangian, the rest is a piece of cake (kinda).

Just work out the ~~[illegible]~~ Lagrange's equations; but first for some preliminaries:

$$L(\theta, \dot\theta) = \frac{1}{2}\dot\theta^T M(\theta)\dot\theta - V(\theta)$$

$$= \frac{1}{2}\sum_{i,j=1}^{n} M_{ij}(\theta)\dot\theta_i\dot\theta_j \quad - V(\theta)$$

Want to find

$$\frac{d}{dt}\frac{\partial L}{\partial \dot\theta_i} - \frac{\partial L}{\partial \theta_i} = \tau_i$$

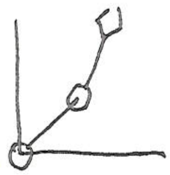
actuator torque & other nonconservative ~~forces~~, generalized forces acting on i^{th} joint.

$$\frac{d}{dt}\frac{\partial L}{\partial \dot\theta_i} = \frac{d}{dt}\left(\sum_{j=1}^{n} M_{ij}\dot\theta_j\right) = \sum_{j=1}^{n}\left(M_{ij}\ddot\theta_j + \dot M_{ij}\dot\theta_j\right)$$

$$= \sum_{j=1}^{n}\left(M_{ij}\ddot\theta_j + \sum_{k=1}^{n}\frac{\partial M_{ij}}{\partial \theta_k}\dot\theta_k\dot\theta_j\right)$$

$$\frac{\partial L}{\partial \theta_i} = \frac{1}{2}\sum_{j,k=1}^{n}\frac{\partial M_{kj}}{\partial \theta_i}\dot\theta_k\dot\theta_j - \frac{\partial V}{\partial \theta_i}$$

Example : Two-Link Planar Manipulator



$$g_{c1}(\theta) = \begin{Bmatrix} \frac{1}{2} l_1 \cos(\theta_1) \\ \frac{1}{2} l_1 \sin(\theta_1) \\ \theta_1 \end{Bmatrix} \qquad g_{c2}(\theta) = \begin{Bmatrix} l_1\cos(\theta_1) + \frac{3}{5} l_2 \cos(\theta_1+\theta_2) \\ l_1 \sin(\theta_1) + \frac{3}{5} l_2 \sin(\theta_1+\theta_2) \\ \theta_1 + \theta_2 \end{Bmatrix}$$

$\Rightarrow$

$$J_{c1}^{body}(\theta) = \begin{bmatrix} 0 & 0 \\ \frac{1}{2} l_1 & 0 \\ 1 & 0 \end{bmatrix} \quad , \quad J_{c2}^{body}(\theta) = \begin{bmatrix} l_1 \sin(\theta_2) & 0 \\ l_1\cos(\theta_2) + \frac{3}{5} l_2 & \frac{3}{5} l_2 \\ 1 & 1 \end{bmatrix}$$

$$L(\theta, \dot\theta) = T(\theta, \dot\theta)$$

• no potential energy

$$T = \frac{1}{2}(\xi_1^{body})^T M_1 \xi_1^{body} + \frac{1}{2}(\xi_2^{body})^T M_2 \xi_2^{body}$$

$$\text{where } M_1 = \begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_1 & 0 \\ 0 & 0 & I_1 \end{bmatrix} \quad \& \quad M_2 = \begin{bmatrix} m_2 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & I_2 \end{bmatrix}$$

$\Rightarrow$

$$T(\theta,\dot\theta) = \frac{1}{2}\dot\theta^T (J_{c1}^{body}(\theta))^T M_1 J_{c1}^{body}(\theta)\dot\theta + \frac{1}{2}\dot\theta^T (J_{c2}^{body}(\theta))^T M_2 J_{c2}^{body}(\theta)\dot\theta$$

$$\frac{1}{2}\dot{\theta}^T \left(J_{c1}^{body}(\theta)\right)^T M_1 J_{c1}^{body}(\theta)\dot{\theta}$$

$$= \frac{1}{2}\dot{\theta}^T \begin{bmatrix} 0 & \frac{1}{2}l_1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_1 & 0 \\ 0 & 0 & I_1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ \frac{1}{2}l_1 & 0 \\ 1 & 0 \end{bmatrix}\dot{\theta}$$

$$= \frac{1}{2}\dot{\theta}^T \begin{bmatrix} 0 & \frac{1}{2}l_1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ \frac{1}{2}m_1 l_1 & 0 \\ I_1 & 0 \end{bmatrix}\dot{\theta}$$

$$= \frac{1}{2}\dot{\theta}^T \begin{bmatrix} \frac{1}{4}m_1 l_1^2 + I_1 & 0 \\ 0 & 0 \end{bmatrix}\dot{\theta}$$

$$\frac{1}{2}\dot{\theta}^T \left(J_{c2}^{body}(\theta)\right)^T M_2 J_{c2}^{body}(\theta)\dot{\theta}$$

$$= \frac{1}{2}\dot{\theta}^T \begin{bmatrix} l_1\sin(\theta_2) & l_1\cos(\theta_2) + \frac{3}{5}l_2 & 1 \\ 0 & \frac{3}{5}l_2 & 1 \end{bmatrix} \begin{bmatrix} m_2 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & I_2 \end{bmatrix} \begin{bmatrix} l_1\sin(\theta_2) & 0 \\ l_1\cos(\theta_2) + \frac{3}{5}l_2 & \frac{3}{5}l_2 \\ 1 & 1 \end{bmatrix}\dot{\theta}$$

$$= \frac{1}{2}\dot{\theta}^T \begin{bmatrix} l_1\sin(\theta_2) & l_1\cos(\theta_2) + \frac{3}{5}l_2 & 1 \\ 0 & \frac{3}{5}l_2 & 1 \end{bmatrix} \begin{bmatrix} m_2 l_1 \sin(\theta_2) & 0 \\ m_2(l_1\cos(\theta_2) + \frac{3}{5}l_2) & \frac{3}{5}m_2 l_2 \\ I_2 & I_2 \end{bmatrix}\dot{\theta}$$

$$= \frac{1}{2}\dot{\theta} \begin{bmatrix} m_2 l_1^2 \sin^2(\theta_2) + m_2\left(l_1\cos(\theta_2) + \frac{3}{5}l_2\right)^2 + I_2 & \frac{3}{5}m_2 l_2 (l_1\cos(\theta_2) + \frac{3}{5}l_2) + I_2 \\ \frac{3}{5}m_2 l_2 (l_1\cos(\theta_2) + \frac{3}{5}l_2) + I_2 & \frac{9}{25}m_2 l_2^2 + I_2 \end{bmatrix}\dot{\theta}$$

$$= \frac{1}{2}\dot{\theta} \begin{bmatrix} m_2 l_1^2 + \frac{9 m_2 l_2^2}{25} + 2\cdot\frac{3}{5}m_2 l_1 l_2 \cos(\theta_2) + I_2 & \frac{3}{5}m_2 l_2 (l_1\cos(\theta_2) + \frac{3}{5}l_2) + I_2 \\ \frac{3}{5}m_2 l_2 (l_1\cos(\theta_2) + \frac{3}{5}l_2) + I_2 & \frac{9}{25}m_2 l_2^2 + I_2 \end{bmatrix}\dot{\theta}$$

$$L(\theta,\dot\theta) = \frac{1}{2}\dot\theta \begin{bmatrix} \frac{1}{4}m_1 l_1^2 + m_2 l_1^2 + \frac{9}{25}m_2 l_2^2 + \frac{6}{5}m_2 l_1 l_2 \cos(\theta_2) + I_1 + I_2 & \frac{3}{5}m_2 l_2 (l_1\cos(\theta_2) + \frac{3}{5}l_2) + I_2 \\ \frac{3}{5}m_2 l_2 (l_1\cos(\theta_2) + \frac{3}{5}l_2) + I_2 & \frac{9}{25}m_2 l_2^2 + I_2 \end{bmatrix} \dot\theta$$

this is $M(\theta)$

now, what about the equations of motion?

just need to figure out:

$$\frac{d}{dt}\frac{\partial L}{\partial\dot\theta} - \frac{\partial L}{\partial\theta} = \tau$$

torques applied to joint actuators.

it will be of the form

$$M(\theta)\ddot\theta + C(\theta,\dot\theta)\dot\theta + \underbrace{N(\theta,\dot\theta)}_{\text{we don't have this part.}} = \tau$$

$$C(\theta,\dot\theta) = \frac{1}{2}\sum_{k=1}^{n}\left(\frac{\partial M_{ij}}{\partial\theta_k} + \frac{\partial M_{ik}}{\partial\theta_j} - \frac{\partial M_{kj}}{\partial\theta_i}\right)\dot\theta_k$$

$$\frac{\partial M}{\partial\theta_1} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \qquad \frac{\partial M}{\partial\theta_2} = \begin{bmatrix} -\frac{6}{5}m_2 l_1 l_2 \sin(\theta_2) & -\frac{3}{5}m_2 l_1 l_2 \sin(\theta_2) \\ -\frac{3}{5}m_2 l_1 l_2 \sin(\theta_2) & 0 \end{bmatrix}$$

$$C_{ij1}(\theta,\dot\theta) = \begin{bmatrix} 0 & -\frac{6}{5}m_2 l_1 l_2 \sin(\theta_2)\dot\theta_1 \\ \frac{6}{5}m_2 l_1 l_2 \sin(\theta_2)\dot\theta_1 & 0 \end{bmatrix}$$

$$C_{ij2}(\theta,\dot\theta) = \begin{bmatrix} -\frac{6}{5}m_2 l_1 l_2 \sin(\theta_2)\dot\theta_2 & -\frac{6}{5}m_2 l_1 l_2 \sin(\theta_2)\dot\theta_2 \\ 0 & 0 \end{bmatrix}$$

$\Rightarrow$

$$C(\theta, \dot{\theta}) = \begin{bmatrix} -\frac{6}{5} m_2 l_1 l_2 \sin(\theta_2) \dot{\theta}_2 & -\frac{6}{5} m_2 l_1 l_2 \sin(\theta_2)(\dot{\theta}_1 + \dot{\theta}_2) \\ \frac{6}{5} m_2 l_1 l_2 \sin(\theta_2) \dot{\theta}_1 & 0 \end{bmatrix}$$

which can be used to obtain the equations of motion as per

$$M(\theta)\ddot{\theta} + C(\theta, \dot{\theta})\dot{\theta} = \tau .$$