Sep. 26 - Oct. 28 (tentative) → 5 weeks.

- forward kinematics → just do product of exponentials for SE(3) manipulator.
- workspace analysis dextrous, reachable
- inverse kinematics (position, velocity)
- trajectory generation w/ splines
  - in joint space
  - in configuration space
  - resolved rate
- potential field methods

Manipulators & Manipulator Analysis

CRAIG CH. 3,4.

- joints.
- Workspace, Configuration Space, Joint Space:

  $$Q = M \times G$$

- Workspace Analysis.

$$\begin{tikzcd}
\text{Actuator Space} \arrow[r, bend right] & \begin{array}{c}\text{Joint Space}\\ \text{(Shape Space)}\end{array} \arrow[l, bend right] \arrow[r, bend right, "\text{forward kinematics}"'] & \begin{array}{c}\text{Configuration Space}\\ \text{(Group Space)}\end{array} \arrow[l, bend right, "\text{inverse kinematics}"']
\end{tikzcd}$$

M G

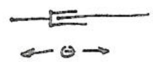
Forward kinematics

- configuration of end-effector given joint configuration of manipulator.
- gives transformation from manipulator base frame to end-effector frame.
- done by concatenating transformations that go from joint-to-joint.

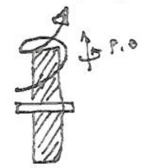
Joints are traditionally chosen from a set of 6 simpler ones, called lower-pair joints,

1) revolute

2) prismatic

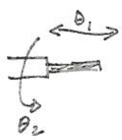


3) helical / screw



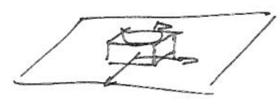
1 d.o.f.

4) cylindrical



5) spherical / ball and socket

6) planar



least common.

- simple nature allows for product of exponentials formula.

if we consider each joint to have 1 d.o.f., then joint space is products of

1) S^1, and (revolute joint)

2) $\mathbb{R}$ (prismatic joint) or helical joint

products of S^1 gives torus, $T^r = \underbrace{S^1 \times \cdots \times S^1}_{r \text{ copies}}$

$$\mathbb{R}^p = \underbrace{\mathbb{R} \times \cdots \times \mathbb{R}}_{p \text{ copies}}$$

$\Rightarrow$

$$M = T^r \times \mathbb{R}^p$$

r - # revolute joints

p - # prismatic/helical joints.

Workspace descriptions:

Workspace (complete)

$$W = \{ g_e(\theta) \in SE(n) \mid \theta \in M \}$$

- set of all configurations reachable by some joint configuration.

* usually difficult to interpret / visualize ↙ alternative

Reachable Workspace

$$W_R = \{ p_e(\theta) \in E(n) \mid \theta \in M \}$$

- set of positions reachable by some joint configuration.

- is a volume of $\mathbb{E}(n)$ which can be reached at <u>some</u> orientation.

* not necessarilly useful measure, since orientation not always controllable.

Dextrous Workspace

$$W_D = \{ p_e(\theta) \in E(n) \mid \forall R \in SO(n), \exists \theta \in M : g_e(\theta) = (p_e, R) \}$$

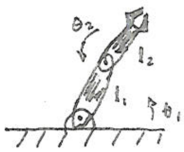
- set of positions reachable with arbitrary orientation.

* within this volume we can do anything.

* typically, to maximize dextrous ~~world~~ workspace, industrial manipulators add a spherical wrist to the end of the manipulator chain.

e.g. SCARA manipulator adds a cylindrical joint for full $SE(2) \subset SE(3)$ control.

Example 1. (Kinematically insufficient manipulator)



$$g_e(\theta) = \begin{Bmatrix} l_1\cos(\theta_1) + l_2\cos(\theta_1+\theta_2) \\ l_1 \sin(\theta_1) + l_2 \sin(\theta_1+\theta_2) \\ \theta_1+\theta_2 \end{Bmatrix}$$

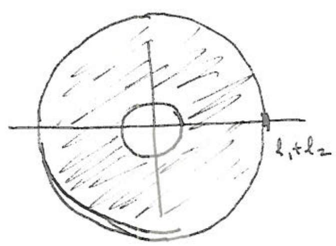
note: $\dim(M) < \dim(G)$

W - will be a surface in SE(3); does not take up any volume.
hard to visualize.

$\downarrow$
system is not fully controllable.

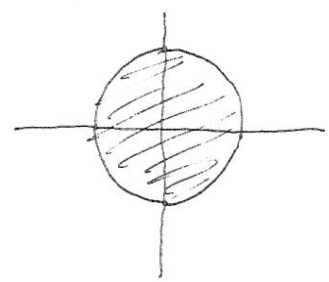
W_R - depends on geometry

if $l_1 \neq l_2$



annulus $|l_1-l_2| < r < l_1+l_2$.

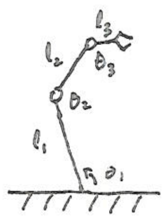
if $l_1 = l_2 = l$



disc of radius $2l$.

W_D - if $l_1 \neq l_2$, get ϕ if $l_1 = l_2 = l$, get origin only.

Example 2.



$$g_e(\theta) = \begin{Bmatrix} l_1\cos(\theta_1) + l_2\cos(\theta_1+\theta_2) + l_3\cos(\theta_1+\theta_2+\theta_3) \\ l_1\sin(\theta_1) + l_2\sin(\theta_1+\theta_2) + l_3\sin(\theta_1+\theta_2+\theta_3) \\ \theta_1+\theta_2+\theta_3 \end{Bmatrix}$$

assume: $l_1 > l_2 > l_3$

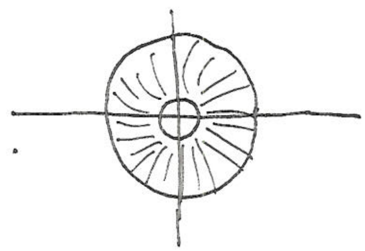
$l_1 > l_2 + l_3$

$\dim(M) = \dim(G)$

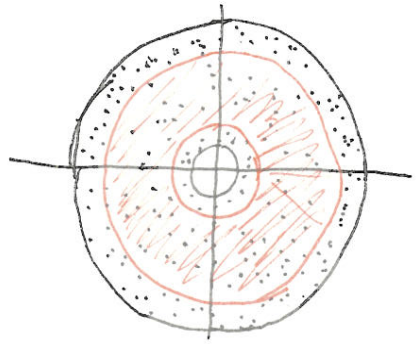
$\Rightarrow$

should have non-trivial dextrous workspace.

W_R - annulus $l_1 - l_2 - l_3 \leq r \leq l_1 + l_2 + l_3$



W_D - annulus $l_1 - l_2 + l_3 \leq r \leq l_1 + l_2 - l_3$



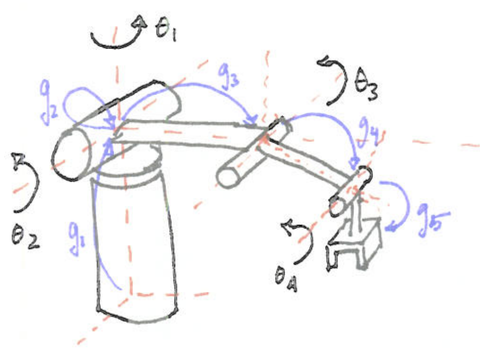
loses $2l_3$ of inner & outer radii.

1) Forward Kinematics

2) Product of Exponentials

Definition. The <u>forward kinematics</u> of a manipulator is the configuration of the end-effector given ~~the relative~~ a joint configuration of the manipulator.

Example:



$$g_e(\vec{\theta}) = g_1(\theta_1)\, g_2(\theta_2)\, g_3(\theta_3)\, g_4(\theta_4)\, g_5$$

for $g = (p, R)$

- getting displacement p_i is easy.

$$p_1 = \begin{Bmatrix} 0 \\ 0 \\ l_0 \end{Bmatrix} \qquad p_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad p_3 = \begin{Bmatrix} 0 \\ l_1 \\ 0 \end{Bmatrix} \qquad p_4 = \begin{Bmatrix} 0 \\ l_2 \\ 0 \end{Bmatrix} \qquad p_5 = \begin{Bmatrix} 0 \\ l_3 \\ 0 \end{Bmatrix}$$

- what about rotations R_i ?

$$R_1 \; \cancel{R_{zz}} = \begin{bmatrix} \cos(\theta_1) & -\sin(\theta_1) & 0 \\ \sin(\theta_1) & \cos(\theta_1) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta_2) & -\sin(\theta_2) \\ 0 & \sin(\theta_2) & \cos(\theta_2) \end{bmatrix}$$

$$R_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta_3) & -\sin(\theta_3) \\ 0 & \sin(\theta_3) & \cos(\theta_3) \end{bmatrix}$$

$$R_4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta_4) & -\sin(\theta_4) \\ 0 & \sin(\theta_4) & \cos(\theta_4) \end{bmatrix} \qquad R_5 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\Rightarrow$

$$g_e = g_1(\theta_1)\, g_2(\theta_2)\, g_3(\theta_3)\, g_4(\theta_4)\, g_5$$

now, $$g_2(\theta_2) g_3(\theta_3) g_4(\theta_4) = \left[\begin{array}{c|c} R_2 R_3 R_4 & p_2 + R_2 p_3 + R_2 R_3 p_4 \\ \hline 0 & 1 \end{array}\right]$$

$$= \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & \cos(\theta_2+\theta_3+\theta_4) & -\sin(\theta_2+\theta_3+\theta_4) & l_1\cos(\theta_2) + l_2\cos(\theta_2+\theta_3) \\ 0 & \sin(\theta_2+\theta_3+\theta_4) & \cos(\theta_2+\theta_3+\theta_4) & l_1\sin(\theta_2) + l_2\sin(\theta_2+\theta_3) \\ \hline 0 & 0 & 0 & 1 \end{array}\right]$$

next,

$$g_2 g_3 g_4 g_5 = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & \cos(\theta_2+\theta_3+\theta_4) & -\sin(\theta_2+\theta_3+\theta_4) & l_1\cos(\theta_2) + l_2\cos(\theta_2+\theta_3) + l_3\sin(\theta_2+\theta_3+\theta_4) \\ 0 & \sin(\theta_2+\theta_3+\theta_4) & \cos(\theta_2+\theta_3+\theta_4) & l_1\sin(\theta_2) + l_2\sin(\theta_2+\theta_3) - l_3\cos(\theta_2+\theta_3+\theta_4) \\ \hline 0 & 0 & 0 & 1 \end{array}\right]$$

$\Rightarrow$

$$g_e = \left[\begin{array}{ccc} \cos(\theta_1) & -\sin(\theta_1)\cos(\theta_2+\theta_3+\theta_4) & \sin(\theta_1)\sin(\theta_2+\theta_3+\theta_4) \\ \sin(\theta_1) & \cos(\theta_1)\cos(\theta_2+\theta_3+\theta_4) & -\cos(\theta_1)\sin(\theta_2+\theta_3+\theta_4) \\ 0 & \sin(\theta_2+\theta_3+\theta_4) & \cos(\theta_2+\theta_3+\theta_4) \\ \hline 0 & 0 & 0 \end{array}\right. \cdots$$

$$\cdots \left.\begin{array}{|c} -\sin(\theta_1)\left(l_1\cos(\theta_2) + l_2\cos(\theta_2+\theta_3) + l_3\sin(\theta_2+\theta_3+\theta_4)\right) \\ \cos(\theta_1)\left(l_1\cos(\theta_2) + l_2\cos(\theta_2+\theta_3) + l_3\sin(\theta_2+\theta_3+\theta_4)\right) \\ l_0 + l_1\sin(\theta_2) + l_2\sin(\theta_2+\theta_3) - l_3\cos(\theta_2+\theta_3+\theta_4) \\ \hline 1 \end{array}\right]$$

Product of Exponentials

Note that $g_e = g_1 \cdot g_2 \cdots g_n \, g_{n+1}$

Is there a way to have $g_e = e^{\hat{\xi}_1 \theta_1} \cdots e^{\hat{\xi}_n \theta_n} g_{n+1}$?

YES. Is it as easy as taking $\hat{\xi}_i = \ln(g_i)$? No

- because g_i's may have constant offsets in them, and we know that $e^{\hat{\xi}_i \cdot 0} = I$, which does nothing.

But, we do know that each $\hat{\xi}_i$ will quantify effect of rotation, translation, or some fixed ratio of both.

Consider trying to find

$$g_e = e^{\hat{\xi}_1 \theta_1} \cdots e^{\hat{\xi}_n \theta_n} g_0 .$$

if $\vec{\theta} = 0$, then

$$g_e(0) = g_0$$

$\Rightarrow$ g_0 is ~~some~~ the reference configuration for the manipulator.

A given $\hat{\xi}_i$ will then say how the reference configuration changes for a given θ_i,

$$g_e(0, \ldots, \theta_i, \ldots, 0) = e^{\hat{\xi}_i \theta_i} g_0$$

what are the ξ_i?

There are three basic types corresponding to the 3 single degree of freedom lower-pair joints

1) revolute: $\xi_i = \begin{Bmatrix} \hat{q}_i \omega_i \\ \omega_i \end{Bmatrix}$ — ω_i - unit vector aligned w/rotation axis; q_i - point on the axis

2) prismatic: $\xi_i = \begin{Bmatrix} v_i \\ 0 \end{Bmatrix}$ — v_i - unit vector aligned w/translation axis.

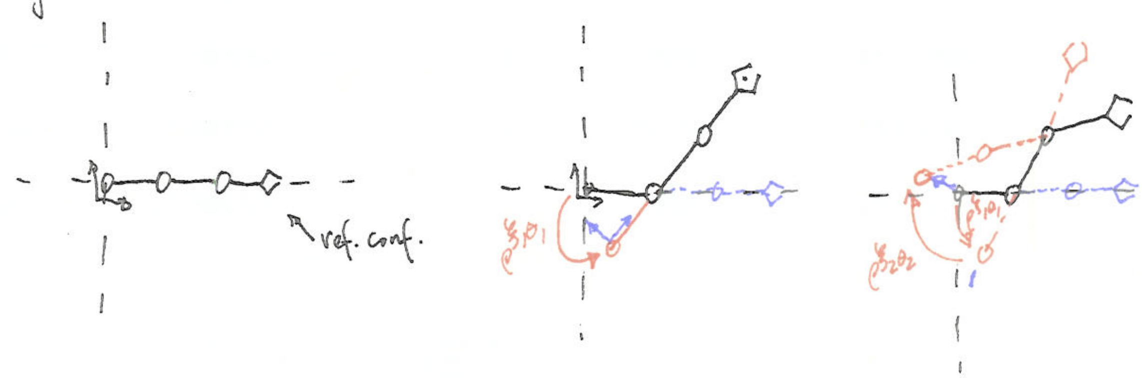
3) helical/screw: $\xi_i = \begin{Bmatrix} h\omega_i + \hat{q}_i \omega_i \\ \omega_i \end{Bmatrix}$ — h - pitch of helical motion.

but what do they mean/represent?

- they represent twists in spatial coordinates, $\xi_i \in \mathfrak{se}(3)$.

* give transformation of reference frame, so that reference configuration ~~take~~ in this new ~~location~~ configuration of reference frame gives new configuration of end-effector frame.

e.g.



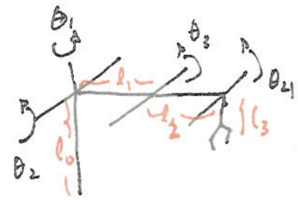
Based on seeing how end-effector changes for each joint from base to end-effector frame, can combine for all i to get total effect:

$$g_e = e^{\hat{\xi}_1 \theta_1} \cdots e^{\hat{\xi}_n \theta_n} g_o$$

order matters !

the $(\hat{\xi}_i, \theta_i)$ must enumerate from base to tool frame.

Example.



$$g_e(\vec{\theta}) = e^{\hat{\xi}_1\theta_1} \cdots e^{\hat{\xi}_4\theta_4}\, g_0$$

$$g_e(\vec{0}) = g_0 = \left[\begin{array}{c|c} I & \begin{Bmatrix} 0 \\ l_1 + l_2 \\ l_0 - l_3 \end{Bmatrix} \\ \hline 0 & 1 \end{array}\right]$$

this is reference configuration

what is ξ_1?

rotation axis $\omega_1 = \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix}$

point on axis is $q_1 = \begin{Bmatrix} 0 \\ 0 \\ l_0 \end{Bmatrix} \Rightarrow q_1 \times \omega_1 = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$

$$\Rightarrow \quad \xi_1 = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{Bmatrix}$$

ξ_2?

$\omega_2 = \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix} \qquad q_2 = \begin{Bmatrix} 0 \\ 0 \\ l_0 \end{Bmatrix} \qquad q_2 \times \omega_2 = \begin{Bmatrix} 0 \\ +l_0 \\ 0 \end{Bmatrix}$

$$\Rightarrow \quad \xi_2 = \begin{Bmatrix} 0 \\ l_0 \\ 0 \\ \hline 1 \\ 0 \\ 0 \end{Bmatrix}$$

ξ_3?

$\omega_3 = \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix} \qquad q_3 = \begin{Bmatrix} 0 \\ l_1 \\ l_0 \end{Bmatrix} \qquad q_3 \times \omega_3 = \begin{Bmatrix} 0 \\ +l_0 \\ -l_1 \end{Bmatrix}$

$$\Rightarrow \quad \xi_3 = \begin{Bmatrix} 0 \\ +l_0 \\ -l_1 \\ 1 \\ 0 \\ 0 \end{Bmatrix}$$

$$\xi_4 ? \qquad \omega_4 = \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix} \qquad q_4 = \begin{Bmatrix} 0 \\ l_1+l_2 \\ l_0 \end{Bmatrix} \qquad q_4 \times \omega_4 = \begin{Bmatrix} 0 \\ l_0 \\ -l_1-l_2 \end{Bmatrix}$$

$$\Rightarrow \quad \xi_3 = \begin{Bmatrix} 0 \\ l_0 \\ -l_1-l_2 \\ 1 \\ 0 \\ 0 \end{Bmatrix}$$

CAN WE VERIFY THIS?

$$g_e = e^{\hat{\xi}_1\theta_1} e^{\hat{\xi}_2\theta_2} e^{\hat{\xi}_3\theta_3} e^{\hat{\xi}_4\theta_4} g_0$$

$$e^{\hat{\xi}_1\theta_1} = \left[\begin{array}{ccc|c} \cos(\theta_1) & -\sin(\theta_1) & 0 & 0 \\ \sin(\theta_1) & \cos(\theta_1) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right]$$

$$e^{\hat{\xi}_2\theta_2} = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & \cos(\theta_2) & -\sin(\theta_2) & l_0\sin(\theta_2) \\ 0 & \sin(\theta_2) & \cos(\theta_2) & l_0(1-\cos(\theta_2)) \\ \hline 0 & 0 & 0 & 1 \end{array}\right]$$

$$e^{\hat{\xi}_3\theta_3} = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & \cos(\theta_3) & -\sin(\theta_2) & l_0\sin(\theta_3) + l_1(1-\cos(\theta_3)) \\ 0 & \sin(\theta_3) & \cos(\theta_2) & l_0(1-\cos(\theta_3)) + l_1\sin(\theta_3) \\ \hline 0 & 0 & 0 & 1 \end{array}\right]$$

$$e^{\hat{\xi}_4 \theta_4} = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & \cos(\theta_4) & -\sin(\theta_4) & l_0 \sin(\theta_4) + (l_1 + l_2)(1 - \cos(\theta_4)) \\ 0 & \sin(\theta_4) & \cos(\theta_4) & l_0(1 - \cos(\theta_4)) - (l_1 + l_2)\sin(\theta_4) \\ \hline 0 & 0 & 0 & 1 \end{array}\right]$$

We get,

$$g_e =$$