Manipulator Jacobian

Ch. 5 of Craig

~~the~~ Manipulator Jacobian – relates joint velocities to end-effector velocities.

recall: $g_e(t) = g_e(\theta(t)) = \prod g_i(\theta(t))$

> Aside on control
>
> $\dot{x} = Ax + Bu$
>
> $x = \begin{Bmatrix} g \\ \theta \end{Bmatrix}$
>
> $\dot{x} = \begin{Bmatrix} \dot{g} \\ \dot{\theta} \end{Bmatrix} = \begin{Bmatrix} J(\theta)\, u \\ u \end{Bmatrix}$
>
> kinematic control (position)

$$J(\theta) = \frac{\partial g_e}{\partial \theta} \quad \text{also written as } Dg_e$$

~~this gives the relation~~ ~~can be used to describe the~~
to relate joint velocities to end-effector velocities:

$$\dot{g}_e = J(\theta) \cdot \dot{\theta}$$

Jacobian is, in general, time-varying and depends on the configuration $\theta(t)$.

To get the joint velocity in body frame:

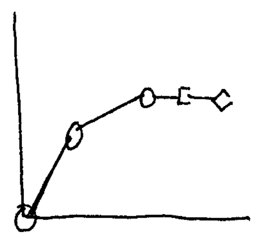
$$\xi_e^b = g_e^{-1} \dot{g}_e = g_e^{-1} J(\theta) \cdot \dot{\theta}$$

and in the spatial frame:

$$\xi_e^s = \dot{g}_e g_e^{-1} = J(\theta) \cdot \dot{\theta}\, g_e^{-1}$$

hybrid Jacobian?

Recall ~~that~~ the manipulator from last time:



$$g_e = \begin{Bmatrix} l_1\cos(\theta_1) + l_2\cos(\theta_1+\theta_2) + l_3\cos(\theta_1+\theta_2+\theta_3) \\ l_1\sin(\theta_1) + l_2\sin(\theta_1+\theta_2) + l_3\sin(\theta_1+\theta_2+\theta_3) \\ \theta_1+\theta_2+\theta_3 \end{Bmatrix}$$

$$J(\theta) = \partial g_e/\partial\theta = \quad -l_1\sin(\theta_1)-l_2\sin(\theta_1+\theta_2)-l_3\sin(\theta_1+\theta_2+\theta_3) \quad -l_2\sin(\theta_1+\theta_2)-l_3\sin(\theta_1+\theta_2+\theta_3) \quad -l_3\sin(\theta_1+\theta_2+\theta_3)$$

$$J(\theta) = \frac{\partial g_e}{\partial\theta} = \begin{bmatrix} -l_1\sin(\theta_1)-l_2\sin(\theta_1+\theta_2)-l_3\sin(\theta_1+\theta_2+\theta_3) & -l_2\sin(\theta_1+\theta_2)-l_3\sin(\theta_1+\theta_2+\theta_3) & -l_3\sin(\theta_1+\theta_2+\theta_3) & \cos(\theta_1+\theta_2+\theta_3) \\ l_1\cos(\theta_1)+l_2\cos(\theta_1+\theta_2)+l_3\cos(\theta_1+\theta_2+\theta_3) & l_2\cos(\theta_1+\theta_2)+l_3\cos(\theta_1+\theta_2+\theta_3) & l_3\cos(\theta_1+\theta_2+\theta_3) & \sin(\theta_1+\theta_2+\theta_3) \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

can use it to tell if we have full control of object at a point, e.g., if we can reach nearby $\boxed{\text{points w/ arbitrary}}$ configurations.

We can take advantage of the fact that this is a product of Lie groups:

$$\dot{g}_e = \dot{g}_1 g_2 g_3 g_4 + g_1 \dot{g}_2 g_3 g_4 + g_1 g_2 \dot{g}_3 g_4 + g_1 g_2 g_3 \dot{g}_4$$

$\Rightarrow$

$$\xi_e^b = g_e^{-1}\dot{g}_e = g_4^{-1} g_3^{-1} g_2^{-1} g_1^{-1}\left[\dot{g}_1 g_2 g_3 g_4 + g_1 \dot{g}_2 g_3 g_4 + g_1 g_2 \dot{g}_3 g_4 + g_1 g_2 g_3 \dot{g}_4\right]$$

$$= Ad^{-1}_{g_2 g_3 g_4}(g_1^{-1}\dot{g}_1) + Ad^{-1}_{g_3 g_4}(g_2^{-1}\dot{g}_2) + Ad^{-1}_{g_4}(g_3^{-1}\dot{g}_3) + g_4^{-1}\dot{g}_4$$

$$= Ad^{-1}_{g_2 g_3 g_4}\left(J_1^b \dot\theta\right) + Ad_{g_3 g_4}\left(J_2^b \dot\theta\right) + Ad_{g_4}\left(J_3^b \dot\theta\right) + J_4^b \dot\theta$$

what are the J_i^b? Well,

$$g_1 = \begin{bmatrix} R(\theta) & 0 \\ 0 & 1 \end{bmatrix} = (0, R(\theta))$$

$$g_1^{-1}\dot g_1 = \begin{bmatrix} \dot\theta_1 J & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow (0, \dot\theta_1)$$

$$J_1^b(\theta)\cdot\dot\theta = \begin{Bmatrix} 0 \\ \dot\theta_1 \end{Bmatrix} \qquad J_1(\theta) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

$$g_2 = \begin{bmatrix} R(\theta_2) & \begin{Bmatrix} l_1 \\ 0 \end{Bmatrix} \\ 0 & 1 \end{bmatrix} \qquad \dot g_2 = \begin{bmatrix} DR\,\dot\theta_2 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow \quad g_2^{-1}\dot g_2 = \begin{bmatrix} \dot\theta_2 J & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow \quad J_2^b(\theta) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$$J_3^b(\theta) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

lastly $$g_4 = \begin{bmatrix} I & \begin{Bmatrix} \theta_4 \\ 0 \end{Bmatrix} \\ 0 & 1 \end{bmatrix} \qquad \dot g_4 = \begin{bmatrix} 0 & \begin{Bmatrix} \dot\theta_4 \\ 0 \end{Bmatrix} \\ 0 & 0 \end{bmatrix}$$

$$g_4^{-1}\dot g_4 = \begin{bmatrix} 0 & \begin{Bmatrix} \dot\theta_4 \\ 0 \end{Bmatrix} \\ 0 & 0 \end{bmatrix} = \begin{Bmatrix} \dot\theta_4 \\ 0 \\ 0 \end{Bmatrix}$$

$$\Rightarrow \quad J_4^b(\theta) = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow \quad \xi_e^b = Ad^{-1}_{g_2g_3g_4} J_1 \dot\theta + Ad_{g_3g_4} J_2\dot\theta + Ad_{g_4}(J_3\dot\theta) + J_4\dot\theta$$

$= (R^{-1}p)^\wedge R^{-1}$

$R^{-1}p \times Rv$

$R^{-1}p^\wedge$

recall that in vector form, $\xi = \begin{Bmatrix} v \\ \omega \end{Bmatrix}$, Adjoint is

$p = -R^{-1}p$

$-(R^{-1}p)^\wedge R$

$$Ad_g \xi = \begin{bmatrix} R & \hat{p}R \\ 0 & R \end{bmatrix} \xi = \begin{bmatrix} R & \hat{p}R \\ 0 & R \end{bmatrix} \begin{Bmatrix} v \\ \omega \end{Bmatrix}$$

$Ad_g^{-1}\xi = Ad_{g^{-1}}\xi = \begin{bmatrix} R^T & -R^T\hat{p} \\ 0 & R^T \end{bmatrix}$

$\Rightarrow$

$$\xi_e^b = \left(Ad^{-1}_{g_2g_3g_4} J_1 + Ad^{-1}_{g_3g_4} J_2 + Ad^{-1}_{g_4} J_3 + J_4 \right) \dot\theta$$

$\Rightarrow$

body manipulator Jacobian

$$J^b = Ad^{-1}_{g_2g_3g_4} J_1 + Ad^{-1}_{g_3g_4} J_2 + Ad^{-1}_{g_4} J_3 + J_4$$

also, spatial manipulator Jacobian

$$J^s = Ad_g J^b = Ad_{g_1g_2g_3g_4} J^b$$

$$= Ad_{g_1} J_1^b + Ad_{g_1g_2} J_2^b + Ad_{g_1g_2g_3} J_3^b + Ad_{g_1g_2g_3g_4} J_4^b$$

$$= J_1^s + Ad_{g_1} J_2^s + Ad_{g_1g_2} J_3^s + Ad_{g_1g_2g_3} J_4^s$$

kinematically redundant → has more than the minimally required degrees of freedom

self-motion manifold - set of joint values which can be used to achieve a desired configuration

internal motions - motions along the self-motion manifold

↳ satisfy

$$J(\theta)\,\dot{\theta} = 0$$

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product of exponents

$$J^s(\theta) = [\,\xi_1 \;\; \xi_2' \;\cdots\; \xi_n'\,]$$

$$\xi_i' = Ad_{(e^{\hat{\xi}_1\theta_1}\cdots e^{\hat{\xi}_{i-1}\theta_{i-1}})}\,\xi_i \quad \text{where} \quad \xi_i = \frac{\partial g}{\partial \theta_i}\, g_i^{-1}$$

$$J^b(\theta) = [\,\xi_1^{+} \;\cdots\; \xi_{n-1}^{+} \;\; \xi_n\,]$$

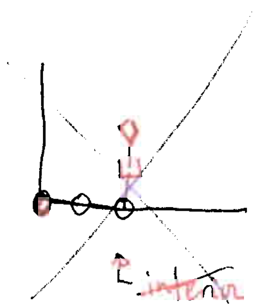
$$\xi_i^{+} = Ad^{-1}_{(e^{\hat{\xi}_{i+1}\theta_{i+1}}\cdots e^{\hat{\xi}_n\theta_n})}\,\xi_i \quad \text{where} \quad \xi_i = g_i^{-1}\,\frac{\partial g}{\partial \theta_i}$$

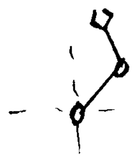
~~Things to say~~

Two kinds of singularities

1) boundary

2) interior





$$g_e = \begin{bmatrix} l_1\cos(\theta_1) + l_2\cos(\theta_1+\theta_2) \\ l_1\sin(\theta_1) + l_2\sin(\theta_1+\theta_2) \\ \theta_1+\theta_2 \end{bmatrix}$$

since $\theta \in \mathbb{R}^2$ can't expect full control

$$J(\theta) = \frac{\partial g_e}{\partial \theta} = \begin{bmatrix} -l_1\sin(\theta_1)-l_2\sin(\theta_1+\theta_2) & -l_2\sin(\theta_1+\theta_2) \\ l_1\cos(\theta_1)+l_2\cos(\theta_1+\theta_2) & l_2\cos(\theta_1+\theta_2) \\ 1 & 1 \end{bmatrix}$$

does not have full row-rank

at $\theta_1 = \frac{\pi}{2}$ $\theta_2 = 0$

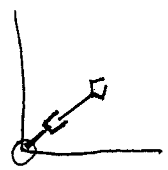
$$J(\tfrac{\pi}{2}, 0) = \begin{bmatrix} -l_1 - l_2 & -l_2 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}$$

in general, let $\theta_2 = 0$

$$J(\theta_1, 0) = \cancel{\begin{matrix} -l_1 - l_2 & -l_2 \\ 0 & \\ 1 & \end{matrix}}$$

$$= \begin{bmatrix} -l_1\sin(\theta_1)-l_2\sin(\theta_1) & -l_2\sin(\theta_1) \\ l_1\cos(\theta_1)+l_2\cos(\theta_1) & l_2\cos(\theta_1) \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -(l_1+l_2)\sin(\theta_1) & -l_2\sin(\theta_1) \\ (l_1+l_2)\cos(\theta_1) & l_2\cos(\theta_1) \\ 1 & 1 \end{bmatrix} \text{parallel vectors}$$



good example

$$g_1 = \begin{bmatrix} I & \begin{Bmatrix} \theta_1 \\ 0 \end{Bmatrix} \\ 0 & 1 \end{bmatrix} \qquad g_2 = \begin{bmatrix} R(\theta_2) & 0 \\ 0 & 1 \end{bmatrix}$$

$$g_3 = \begin{bmatrix} I & \begin{Bmatrix} \theta_3 \\ 0 \end{Bmatrix} \\ 0 & 1 \end{bmatrix}$$

$$g_e = g_1 g_2 g_3 = \begin{bmatrix} I & \begin{Bmatrix} \theta_1 \\ 0 \end{Bmatrix} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} R(\theta_2) & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} I & \begin{Bmatrix} \theta_3 \\ 0 \end{Bmatrix} \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} R(\theta_2) & \begin{Bmatrix} \theta_1 \\ 0 \end{Bmatrix} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} I & \begin{Bmatrix} \theta_3 \\ 0 \end{Bmatrix} \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} R(\theta_2) & \begin{Bmatrix} \theta_1 \\ 0 \end{Bmatrix} + R(\theta_2) \begin{Bmatrix} \theta_3 \\ 0 \end{Bmatrix} \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} R(\theta_2) & \begin{Bmatrix} \theta_1 + \theta_3 \cos(\theta_2) \\ \theta_3 \sin(\theta_2) \end{Bmatrix} \\ 0 & 1 \end{bmatrix} = \begin{Bmatrix} \theta_1 + \theta_3 \cos(\theta_2) \\ \theta_3 \sin(\theta_2) \\ \theta_2 \end{Bmatrix}$$

$$\dot{g}_e = \quad DR(\theta_2)\dot{\theta}_2 \qquad \dot{\theta}_1 + \dot{\theta}_3 \cos(\theta_2) +$$

$$0$$

$$\dot{g}_e = \begin{bmatrix} 1 & -\theta_3 \sin(\theta_2) & \cos(\theta_2) \\ 0 & \theta_3 \cos(\theta_2) & \sin(\theta_2) \\ 0 & 1 & 0 \end{bmatrix} \dot{\theta}$$

$$J(1,0,1) = \underbrace{\begin{bmatrix} 1 & 0 & 1 \\ 0 & \theta_3 & 0 \\ 0 & 1 & 0 \end{bmatrix}}_{\text{loses rank.}} \dot{\theta} \qquad J(1, \tfrac{\pi}{2}, 1) = \begin{bmatrix} 1 & -\theta_3 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

Definition. A <u>singular configuration</u> or <u>singularity</u> is a joint-configuration of an open-chain manipulator in which the end-effector instantaneously loses a degree of freedom of its motion capability versus the number of degrees of freedom that normally prevail (i.e. end-effector loses its ability to move in one or more directions).

Practical Implications:

1) manipulator loses effectiveness
2) high joint velocities may be needed near a singular configuration in order to track a specified trajectory.
3) the manipulator has a high mechanical advantage in a singular configuration.

Finding Singular Configurations

1) Numerical approaches

- $V = J(\theta)\dot{\theta}$ or Jacobian loses rank.

2) for non-redundant & kinematically sufficient arms,

$$\det(J(\theta)) = 0$$

at a singular configuration.
Find all θ^* such that determinant vanishes.

3) for redundant arms, $J(\theta)J^T(\theta)$ has same rank as $J(\theta)$, but is square. Find all θ^* such that

$$\det(J(\theta)J^T(\theta)) = 0$$

4) for kinematically insufficient arms $J^T(\theta)J(\theta)$ has same rank as $J(\theta)$, but is square. Find all θ^* such that $\det(J^T(\theta)J(\theta)) = 0$.

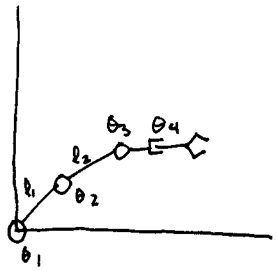
Loss of control → lose rank in the Jacobian

this is called a singularity

Two-types:

1) Workspace-boundary singularities
   - when manipulator is fully stretched out or folded back on itself in such a way that the end-effector is at or near the workspace boundary

2) Workspace-interior singularities
   - occur away from the workspace boundary; generally caused by a lining up of two or more axes.



$$g_e = \begin{Bmatrix} l_1\cos(\theta_1) + l_2\cos(\theta_1+\theta_2) + l_3\cos(\theta_1+\theta_2+\theta_3) \\ l_1\sin(\theta_1) + l_2\sin(\theta_1+\theta_2) + l_3\sin(\theta_1+\theta_2+\theta_3) \\ \theta_1+\theta_2+\theta_3 \end{Bmatrix}$$

$$J(\theta) = \frac{\partial g_e}{\partial \theta} = \begin{bmatrix} -l_1\sin(\theta_1) - l_2\sin(\theta_1+\theta_2) - l_3\sin(\theta_1+\theta_2+\theta_3) & -l_2\sin(\theta_1+\theta_2) - l_3\sin(\theta_1+\theta_2+\theta_3) & -l_3\sin(\theta_1+\theta_2+\theta_3) & \cos(\theta_1+\theta_2+\theta_3) \\ l_1\cos(\theta_1) + l_2\cos(\theta_1+\theta_2) + l_3\cos(\theta_1+\theta_2+\theta_3) & l_2\cos(\theta_1+\theta_2) + l_3\cos(\theta_1+\theta_2+\theta_3) & l_3\cos(\theta_1+\theta_2+\theta_3) & \sin(\theta_1+\theta_2+\theta_3) \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

- can use it to tell if we have full control of object at a point, e.g., if we can reach arbitrary configurations nearby.

$J(\theta)$ for $\theta_0 = (0, 0, 0, \theta_4)$ is

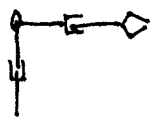
$$J(\theta) = \begin{bmatrix} 0 & 0 & 0 & 1 \\ l_1+l_2+l_3 & l_2+l_3 & l_3 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

has full row-rank.

but, what if θ_4 is extended/retracted as far as possible?

$\Rightarrow$ last option is restricted/not possible

$\Rightarrow$ actuator limits cause a loss in rank (workspace boundary singularity)



$$g_e = g_1 g_2 g_3$$

$$= \begin{bmatrix} I & \begin{Bmatrix} 0 \\ \theta_1 \end{Bmatrix} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} R(\theta_2) & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} I & \begin{Bmatrix} \theta_3 \\ 0 \end{Bmatrix} \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} R(\theta_2) & \begin{Bmatrix} 0 \\ \theta_1 \end{Bmatrix} + R(\theta_2) \begin{Bmatrix} \theta_3 \\ 0 \end{Bmatrix} \\ 0 & 1 \end{bmatrix}$$

$$= \begin{Bmatrix} \theta_3 \cos(\theta_2) \\ \theta_1 + \theta_3 \sin(\theta_2) \\ \theta_2 \end{Bmatrix}$$

$$J(\theta) = \frac{\partial g_e}{\partial \theta} = \begin{bmatrix} 0 & -\theta_3 \sin(\theta_2) & \cos(\theta_2) \\ 1 & \theta_3 \cos(\theta_2) & \sin(\theta_2) \\ 0 & 1 & 0 \end{bmatrix}$$

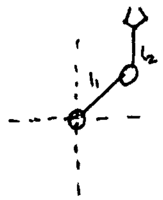
at $\theta_0 = (l_1, 0, l_2)$, $J(\theta) = \begin{bmatrix} 0 & 0 & 1 \\ 1 & l_2 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ ← can change to nearby configurations no problem. (important NEARBY)

rank is 3

at $\theta_2 = (l_1, \frac{\pi}{2}, l_2)$, $J(\theta) = \begin{bmatrix} 0 & -l_2 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ ← lose local control authority. can you see why physically?

rank is 2

↑ workspace interior singularity



$$g_e = \left\{\begin{array}{c} l_1\cos(\theta_1) + l_2\cos(\theta_1+\theta_2) \\ l_1\sin(\theta_1) + l_2\sin(\theta_1+\theta_2) \\ \hline \theta_1+\theta_2 \end{array}\right\}$$

perhaps only consider position control only.

since $\theta \in \mathbb{R}^2$, can't expect full control.

$$J(\theta) = \frac{\partial g_e}{\partial \theta} = \left[\begin{array}{cc} -l_1\sin(\theta_1) - l_2\sin(\theta_1+\theta_2) & -l_2\sin(\theta_1+\theta_2) \\ l_1\cos(\theta_1) + l_2\cos(\theta_1+\theta_2) & l_2\cos(\theta_1+\theta_2) \\ \hline 1 & 1 \end{array}\right]$$

does not have full row-rank.

at $\theta_0 = (\theta_1, 0)$,

$$J(\theta_0) = \left[\begin{array}{cc} -l_1\sin(\theta_1) - l_2\sin(\theta_1) & -l_2\sin(\theta_1) \\ l_1\cos(\theta_1) + l_2\cos(\theta_1) & l_2\cos(\theta_1) \\ \hline 1 & 1 \end{array}\right]$$

$$= \left[\begin{array}{cc} -(l_1+l_2)\sin(\theta_1) & -l_2\sin(\theta_1) \\ (l_1+l_2)\cos(\theta_2) & l_2\cos(\theta_1) \\ \hline 1 & 1 \end{array}\right]$$

parallel vectors

← same

$\Rightarrow$

$J(\theta_0)$ has lost rank.