Inverse Kinematics

Craig (Ch. 4)

- determining a feasible joint-configuration given a desired end-effector configuration.

  - this is a nonlinear problem.
  - clearly desired end-effector configuration must be in workspace to have a solution.

    sometimes will have multiple solutions.

- no real general algorithms exist to handle the nonlinear inverse kinematics problem.

- problem/manipulator considered solveable if joint-configuration can be determined by an algorithm that allows one to determine all joint variable associated w/a given end-effector configuration.

closed-form solutions → algebraic, geometric, Paden-Kahan

Pieper's Solution When Three Axes Intersect.

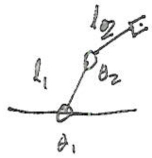
Main approaches:

geometric	easy to understand/apply	not systematic not always applicable
algebraic	systematic & general	hard to understand hard to use
Paden-Kahan	easy to understand systematic	not general.

The Geometric Approach

- by observation, reduce geometry of the linkage to easily solved sub-problems.
  - one technique reduces the linkage geometry to triangles.

example. many manipulators ~~can be reduced~~ have the following subproblem, which is usually obtained by projecting the manipulator geometry to 2 coordinate axes.

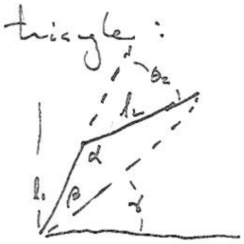


$$\begin{Bmatrix} x_e \\ y_e \\ \theta_e \end{Bmatrix} = \begin{Bmatrix} l_1 \cos\theta_1 + l_2 \cos(\theta_1+\theta_2) \\ l_1 \sin\theta_1 + l_2 \sin(\theta_1+\theta_2) \\ \theta_1 + \theta_2 \end{Bmatrix}$$

orientation is not fully controllable, but consider placement of end-effector in Euclidean space.

the manipulator (part) forms a triangle:

$\sqrt{x_e^2 + y_e^2} = r \longrightarrow$



$\theta_2 = \pi - \alpha$

$\cos(\alpha) = \dfrac{l_1^2 + l_2^2 - r^2}{2 l_1 l_2}$

(law of cosines)

CHECK: $l_1 + l_2 < \sqrt{x_e^2 + y_c^2}$

$\theta_1 = \beta + \gamma$

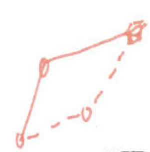
$\gamma = \mathrm{atan}(x_e, y_e)$

$\cos(\beta) = \dfrac{l_1^2 + r^2 - l_2^2}{2 l_1 r}$

$\Rightarrow$

$\theta_1 = \gamma + \beta, \ \gamma - \beta$

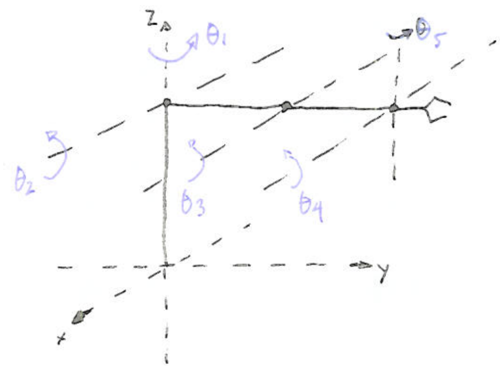
$\theta_2 = \pi - \alpha, \ \pi + \alpha$



check.

orientation: $\theta_1 + \theta_2 = \gamma + \beta + \pi - \alpha, \ \gamma - \beta + \pi + \alpha$?

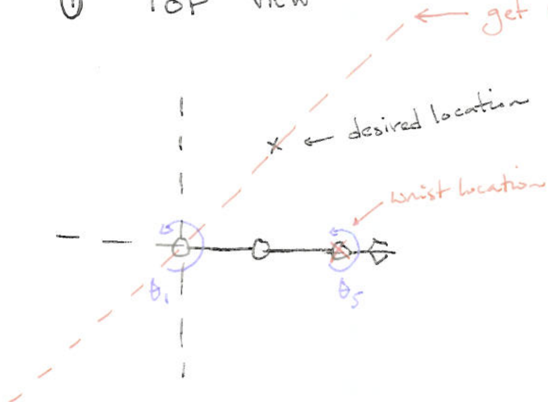
How can geometric techniques be applied to more complicated manipulators.



1) has wrist at end for reorientation

2) to place wrist at a particular location, consider the following projections

① TOP VIEW



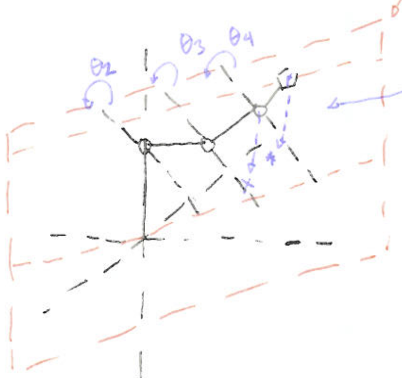
get manipulator to lie along this line. then we can project to this plane since all of the remaining joints work in this plane.

basically, $\theta_1 = \text{atan}(x, y) - \frac{\pi}{2}$.

since reference configuration aligned w/ y-axis & not x-axis.

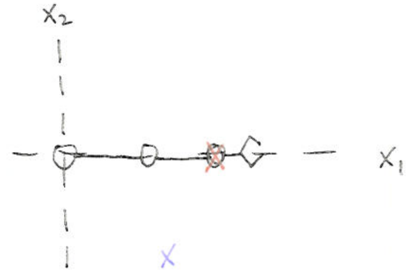
② SIDE VIEW

plane of projection



end-effector to high and to far out. (same with wrist)

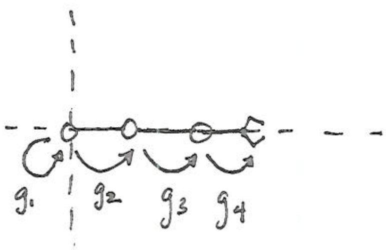
when projected to the plane



desired wrist location

placement of wrist reduces to previous subproblem.
(θ_2, θ_3)

or, consider the following



here, we can control orientation inside the dextrous workspace

(discussed this before)

have desired end-effector configuration g_e^*

actual $g_e(\vec{\theta}) = g_1(\theta_1)\, g_2(\theta_2)\, g_3(\theta_3)\, g_4$

recall each $g_i(\theta_i) = \bar{g}_i\, \tilde{g}_i(\theta_i)$

$\tilde{g}_i$: varies w/ θ_i.

$\bar{g}_i$: constant.

$\Rightarrow$

$$g_e = g_1 g_2 \bar{g}_3 \tilde{g}_3 g_4$$

in our case, this is just rotation, no translation.

so, we desire a joint configuration $\vec{\theta}^*$ such that

$$g_e^* = g_e(\vec{\theta}^*)$$

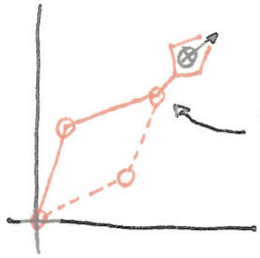
$$\Rightarrow \quad g_e^* = g_1 g_2 \bar{g}_3 \tilde{g}_3 g_4$$

$$\Rightarrow \quad g_h^* = g_e^* g_4^{-1} \tilde{g}_3^{-1} = g_1 g_2 \bar{g}_3$$

last revolute joint is wrist.
can achieve arbitrary reorientation

$\Rightarrow$

only care about position of $g_1 g_2 \bar{g}_3$.



just need to find θ_1, θ_2 to position

$$g_1 g_2 \bar{g}_3 \quad \text{at} \quad (x_n, y_n)$$

but this is our subproblem, the one we've solved before.

once we have θ_1 and θ_2, what is θ_3?

well, $\quad \theta_1 + \theta_2 + \theta_3 = \theta^*$

$\Rightarrow$

$$\theta_3 = \theta^* - \theta_1 - \theta_2$$

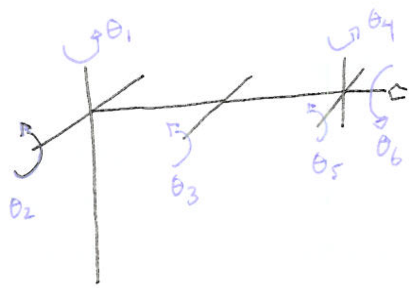
about the geometric approach,

① it's easy because you are working it out intuitively using geometry.

② not systematic because requires good intuition.

③ not always applicable (limited set of "geometric techniques")
- good thing is that design of manipulator can allow for simple geometric solution.

Elbow Manipulator. (Algebraic Method)



$$g_e = g_1 g_2 g_3 g_4 g_5 g_6 \underset{\uparrow\ \text{constant}}{g_7}$$

go to wrist:

$$g_e = g_1 g_2 g_3 \bar{g}_4 \tilde{g}_4 g_5 g_6 g_7$$

$$g_e^* = \underbrace{g_1 g_2 g_3 \bar{g}_4}_{\text{can position anywhere w/in the dextrous workspace, but have minimal to no control over orientation.}} \overbrace{\tilde{g}_4 g_5 g_6}^{\text{allows for arbitrary reorientation. (no translation)}} g_7$$

$\Rightarrow$ define

$g_n^* = g_e^* g_7^{-1}$ recall that position of $g_e^* g_7^{-1}$

same as position of $g_e^* g_7^{-1} g_6^{-1} g_5^{-1} \tilde{g}_4^{-1}$.

want to solve for $(\theta_1, \theta_2, \theta_3)$ such that

$g_w(\vec{\theta}) = g_1(\theta_1) g_2(\theta_2) g_3(\theta_3) \bar{g}_4$ positions wrist at g_n^* position.

e.g. $\text{position}(g_w(\vec{\theta})) = \text{position}(g_n^*(\vec{\theta}))$

- this approach is similar to Pieper's solution when 3-axes intersect.

elbow manipulator wrist position

$$x_w = -(l_1 \cos(\theta_2) + l_2 \cos(\theta_2+\theta_3)) \sin(\theta_1)$$

$$y_w = (l_1 \cos(\theta_2) + l_2 \cos(\theta_2+\theta_3)) \cos(\theta_1)$$

$$z_w = \boxed{l_0} + l_1 \sin(\theta_2) + l_2 \sin(\theta_2+\theta_3)$$

can remove by translating origin up. will adjust the coordinates of the desired wrist point.

Algebraic approach:

- try to solve the simultaneous equations
- elimination theory
  - successively eliminate variables until a single equation in one unknown is reached.
  - for manipulator forward kinematics equations, there exists a systematic approach to carry out this procedure.
  - back-substitution gives solution.

$$x_w^2 + y_w^2 + z_w^2 = l_1^2 + l_2^2 + 2 l_1 l_2 \cos(\theta_3)$$

$\Rightarrow$

$$\cos(\theta_3) = \frac{x_w^2 + y_w^2 + z_w^2 - l_1^2 - l_2^2}{2 l_1 l_2}$$

- two solutions to this problem.

next,

$$x_w^2 + y_w^2 = (l_1 \cos(\theta_2) + l_2 \cos(\theta_2 + \theta_3))^2$$

$\Rightarrow$

$$r = \sqrt{x_w^2 + y_w^2} = l_1 \cos(\theta_2) + l_2 \cos(\theta_2 + \theta_3)$$

$\Rightarrow$

$$r = (l_1 + l_2 \cos(\theta_3)) \cos(\theta_2) - (l_2 \sin(\theta_3)) \sin(\theta_2)$$

if define $w = \tan(\frac{1}{2}\theta_2)$, then $\cos(\theta_2) = \frac{1-w^2}{1+w^2}$

$$\sin(\theta_2) = \frac{2w}{1+w^2}$$

$\Rightarrow$

$$r = (l_1 + l_2 \cos(\theta_3)) \frac{1-w^2}{1+w^2} - (2 l_2 \sin(\theta_3)) \frac{w}{1+w^2}$$

$\Rightarrow$

$$(1+w^2) r = (l_1 + l_2 \cos(\theta_3))(1-w^2) - (2 l_2 \sin(\theta_3)) w$$

$\Rightarrow$

$$(r + l_1 + l_2 \cos(\theta_3)) w^2 + (2 l_2 \sin(\theta_3)) w + (r - l_1 - l_2 \cos(\theta_3)) = 0$$

quadratic formula, can be solved.

given θ_3, get solution(s) for w and find θ_2 using $\tan(\frac{1}{2}\theta_2) = w$.

done for each solution of θ_3

$\Rightarrow$ now there are four possible solutions.

Lastly, to get θ_1

$$\sin(\theta_1) = -\frac{x_w}{l_1\cos(\theta_2) + l_2\cos(\theta_2+\theta_3)}$$

$$\cos(\theta_1) = \frac{y_w}{l_1\cos(\theta_2) + l_2\cos(\theta_2+\theta_3)}$$

$\Rightarrow$

$$\theta_1 = \text{atan2}(\cos(\theta_1), \sin(\theta_1))$$

$$\theta_1 = \text{atan2}(y_w, -x_w)$$ ← only 1 solution.

* an equivalent solution should be $\text{atan2}(x_w, y_w) - \frac{\pi}{2}$.

* must check all four solutions for validity.
  of the valid solutions, just pick one of them.

* once positioned, the goal is to reorient the ~~e~~ end-effector frame.

$$g_e^* = \underbrace{g_1 g_2 g_3 \bar{g}_4}\ \tilde{g}_4 g_5 g_6 g_7$$

$$g_e^* = g_w\ \tilde{g}_4 g_5 g_6 g_7$$

$\Rightarrow$

$$g_w^{-1} g_e^* = \tilde{g}_4 g_5 g_6 g_7$$

$\Rightarrow$

$$g_w^{-1} g_e^* = g_h(\vec{\theta}) \equiv \overbrace{\tilde{g}_4(\theta_4)\, g_5(\theta_5)\, g_6(\theta_6)\, g_7}^{\text{now find these joint angles.}}$$

$\Rightarrow$

$$g_h^* = g_w^{-1} g_e^* g_7^{-1} = g_h(\vec{\theta})$$

$\Rightarrow$ solve for ...

orientation of g_h^* = orientation of $g_h(\vec{\theta})$