Exponential Representation of Group Motions / Displacements.

1) $SO(3)$ and $\mathfrak{so}(3)$
2) $SE(3)$ and $\mathfrak{se}(3)$

→ exp, ln & their properties.

consider a point on the body undergoing rotation

$$\dot{q}(t) = \omega \times q(t) = \hat{\omega}\, q(t) \quad , \quad \cancel{q(0) = q_0}$$

ω ↑ velocity of rotation of axes.

⇒

$$q(t) = e^{\hat{\omega} t} q(0)$$

$$\hookrightarrow \quad e^{\hat{\omega} t} = \sum_{0}^{\infty} \frac{1}{n!} (\hat{\omega} t)^n$$

if rotated for θ units of time,

$$R(\omega, \theta) = e^{\hat{\omega}\theta}$$

need identity for triple cross-product

$\hat{a}^2 = \hat{a}\hat{a}$

$\hat{a}^2 q = \hat{a}\hat{a} q$

$= a \times (a \times q)$

$=$

what is $e^{\hat{\omega}\theta}$?

let's do it for $\mathfrak{so}(3)$ since $\mathfrak{so}(2)$ is just special case of this.

Lemma. Given $a \in \mathbb{R}^3$ such that $\hat{a} \in \mathfrak{so}(3)$, then

$$\hat{a}^2 = aa^T - \|a\|^2 I$$

$$\hat{a}^3 = -\|a\|^2 \hat{a}.$$

From these identities, higher powers of $\hat{a}$ can be found

e.g. $\hat{a}^4 = \hat{a}^3\hat{a} = -\|a\|^2 \hat{a}\hat{a} = -\|a\|^2 aa^T + \|a\|^4 I$

$\qquad = -\|a\|^2 \hat{a}^2$

$\hat{a}^{2k+2} = (-1)^{k+1} \|a\|^{2k} \hat{a}^2$

$\hat{a}^5 = \|a\|^4 \hat{a} \qquad \hat{a}^6 = -\|a\|^4 \hat{a}^2 \quad \ldots \quad \hat{a}^{2k+1} = (-1)^{k+1} \|a\|^{2k} \hat{a}$

Suppose that $a = \omega\theta$ and $\|\omega\| = 1$, then

$$e^{\hat{\omega}\theta} = \sum_0^\infty \frac{1}{n!}(\hat{\omega}\theta)^n = \sum_0^\infty \frac{1}{2k!}(\hat{\omega}\theta)^{2k} + \sum_0^\infty \frac{1}{(2k+1)!}(\hat{\omega}\theta)^{2k+1}$$

$$= I + \sum_1^\infty \frac{1}{2k!}(-\theta)^{k+1}\hat{\omega}^2 + \sum_0^\infty \frac{1}{(2k+1)!}(-\theta)^k \hat{\omega}$$

$$= \underline{I + \hat{\omega}\sin(\theta) + \hat{\omega}^2(1-\cos(\theta))} \rightarrow \text{Rodrigues' formula.}$$

if $\|\omega\| \neq 1$, then

$$e^{\hat{\omega}\theta} = I + \frac{\hat{\omega}}{\|\omega\|}\sin(\|\omega\|\theta) + \frac{\hat{\omega}^2}{\|\omega\|}(1-\cos(\|\omega\|\theta))$$

Need to show that $e^{\hat{\omega}\theta} \in SO(3)$.

how? $\quad R^T R = I$ and $\det(R) = 1$

$$(e^{\hat{\omega}\theta})^T e^{\hat{\omega}\theta} = e^{\hat{\omega}^T\theta}e^{\hat{\omega}\theta} = e^{-\hat{\omega}\theta}e^{\hat{\omega}\theta}$$

$$= e^{-\hat{\omega}\theta + \hat{\omega}\theta} = e^0 = I$$

$$\Rightarrow (e^{\hat{\omega}\theta})^{-1} = (e^{\hat{\omega}\theta})^T$$

$$\det(e^{\hat{\omega}\theta}) = e^{\mathrm{Tr}(\hat{\omega}\theta)} = e^0 = 1$$

What about the other way around?

Proposition. The exponential map is surjective onto $SO(3)$. Given $R \in SO(3)$, there exists an $\omega \in \mathbb{R}^3$, $\|\omega\|=1$ and $\theta \in \mathbb{R}$ such that $R = \exp(\hat{\omega}\theta)$.

note that this is equivalent to saying that $\exists$ a function taking in an R and returning an α such that

$$R = e^{\alpha}$$

this function will be denoted by $\ln$,

$$\alpha = \ln R$$

(α is not necessarily unique!)

proof is nasty & constructive:

$$R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{32} \\ r_{23} & r_{23} & r_{33} \end{bmatrix}$$

only 3 of these are unique

$\hat{\omega}\theta$ has only 3 unique values also.

$$e^{\hat{\omega}\theta} = I + \hat{\omega}\sin\theta + \hat{\omega}^2(1-\cos\theta)$$

$$= \begin{bmatrix} 1-v_\theta(\omega_2^2+\omega_3^2) & \omega_1\omega_2 v_\theta - \omega_3 s_\theta & \omega_1\omega_3 v_\theta + \omega_2 s_\theta \\ \omega_1\omega_2 v_\theta + \omega_3 s_\theta & 1-v_\theta(\omega_1^2+\omega_3^2) & \omega_2\omega_3 v_\theta - \omega_1 s_\theta \\ \omega_1\omega_3 v_\theta - \omega_2 s_\theta & \omega_2\omega_3 v_\theta + \omega_1 s_\theta & 1-v_\theta(\omega_1^2+\omega_2^2) \end{bmatrix}$$

$$= \begin{bmatrix} \omega_1^2 v_\theta + c_\theta & \omega_1\omega_2 v_\theta - \omega_3 s_\theta & \omega_1\omega_3 v_\theta + \omega_2 s_\theta \\ \omega_1\omega_2 v_\theta + \omega_3 s_\theta & \omega_2^2 v_\theta + c_\theta & \omega_2\omega_3 v_\theta - \omega_1 s_\theta \\ \omega_1\omega_3 v_\theta - \omega_2 s_\theta & \omega_2\omega_3 v_\theta + \omega_1 s_\theta & \omega_3^2 v_\theta + c_\theta \end{bmatrix}$$

equating matrices we get

$$\text{trace}(R) = r_{11}+r_{22}+r_{33} = 1+2\cos\theta$$

now $\text{trace}(R) = \lambda_1+\lambda_2+\lambda_3$ ← eigenvalues of R → $1, \lambda, \lambda^*$

$= 1 + 2\text{Re}(\lambda)$

$\det(R) = \Pi\lambda_i = \lambda\cdot\lambda^* = 1 \Rightarrow |\lambda| = 1$

$\det(R) = +1 \Rightarrow |\lambda| = 1$

$\Rightarrow$

$$1 + 2\text{Re}(\lambda) = 1+2\cos\theta$$

$\Rightarrow$

$$\theta = \cos^{-1}\left(\frac{\text{trace}(R)-1}{2}\right)$$

can go $\pm$ and $\pm 2\pi n$

equating off diagonal terms,

$$r_{32}-r_{23} = 2\omega_1 s_\theta$$
$$r_{13}-r_{31} = 2\omega_2 s_\theta$$
$$r_{21}-r_{12} = 2\omega_3 s_\theta$$

if $\theta \neq 0$, choose,

$$\omega = \frac{1}{2s_\theta}\begin{Bmatrix} r_{32}-r_{23} \\ r_{13}-r_{31} \\ r_{21}-r_{12} \end{Bmatrix}$$

or $2\pi-\theta$, $-\omega$

if $\theta = 0$, then ω is arbitrary

$\omega\theta \in \mathbb{R}^3$ are called the exponential coordinates for R.

To recap:

Given $\omega \in \mathbb{R}^3$ s.t. $\hat{\omega} \in so(3)$,

$$e^{\hat{\omega}\theta} = I + \frac{\hat{\omega}}{\|\omega\|}\sin(\|\omega\|\theta) + \frac{\hat{\omega}^2}{\|\omega\|}(1-\cos(\|\omega\|\theta)) \in SO(3)$$

Or, given $R \in SO(3)$, $\exists\, \omega \in \mathbb{R}^3, \theta \in \mathbb{R}$ defined by

$$\theta = \cos^{-1}\left(\frac{\text{trace}(R)-1}{2}\right)$$

if $\theta = 0$, ω is arbitrary

$\theta \neq 0$, $\omega = \dfrac{1}{2\sin(\theta)}\begin{bmatrix} r_{32} - r_{23} \\ r_{13} - r_{31} \\ r_{21} - r_{12} \end{bmatrix}$

in fact,

$$\hat{\omega}\theta = \ln R$$

Can we do the same for $SE(3)$ & $se(3)$?

$$\xi = \begin{Bmatrix} v \\ \omega \end{Bmatrix} \qquad \hat{\xi} = \left[\begin{array}{c|c} \hat{\omega} & v \\ \hline 0 & 0 \end{array}\right]$$

may or may not see the opposite $\left[\begin{array}{c|c} \hat{\omega} & v \\ \hline 0 & 0 \end{array}\right]^{\vee} = \begin{Bmatrix} v \\ \omega \end{Bmatrix}$

$$\hat{\xi}^{\vee} = \xi$$

$$e^{\hat{\xi}\theta} \in SE(3).$$

$$e^{\hat{\xi}\theta} = \sum_{0}^{\infty} \frac{1}{n!}(\hat{\xi}\theta)^n$$

what is $\hat{\xi}^n$?

$$\hat{\xi}^1 = \left[\begin{array}{c|c} \hat{\omega} & v \\ \hline 0 & 0 \end{array}\right]$$

$$\hat{\xi}^2 = \begin{bmatrix} \hat{\omega}^2 & \hat{\omega}v \\ 0 & 0 \end{bmatrix}$$

$$\hat{\xi}^n = \left[\begin{array}{c|c} \hat{\omega}^n & \hat{\omega}^{n-1}v \\ \hline 0 & 0 \end{array}\right]$$

$\Rightarrow$ if $\omega = 0$, then $\hat{\xi}^n = 0$ for $n \geq 2$.

$$e^{\hat{\xi}\theta} = \left[\begin{array}{c|c} I & v\theta \\ \hline 0 & 1 \end{array}\right] \quad \text{for } \omega = 0$$

$\hat{\omega}^2 = \omega\omega^T - \|\omega\|^2 I$

$\hat{\omega}^3 = -\|a\|^2 \hat{a}$

• this is definitely in $SE(3)$.

else $\cancel{e^{\hat{\xi}\theta} = I + \sum_{1}^{\infty} \frac{1}{n!}\left[\begin{array}{c|c} \hat{\omega}^n & \hat{\omega}^{n-1}v \\ \hline 0 & 0 \end{array}\right]\theta^n}$ OK?

$$\cancel{= \left[\begin{array}{c|c} e^{\hat{\omega}\theta} & \sum_{1}^{\infty} \frac{1}{n!}\hat{\omega}^{n-1}v\,\theta^n \\ \hline 0 & 1 \end{array}\right]}$$

SE(3) and $\mathfrak{se}(3)$:

$\xi \in \mathfrak{se}(3)$, $\xi = \begin{Bmatrix} v \\ \omega \end{Bmatrix}$

$$\hat{\xi} = \left[\begin{array}{c|c} \hat{\omega} & v \\ \hline 0 & 0 \end{array}\right]$$

what is $e^{\hat{\xi}\tau}$?

(a) well, if $\omega = \vec{0}$, then $\hat{\xi} = \left[\begin{array}{c|c} 0 & v \\ \hline 0 & 0 \end{array}\right] \Rightarrow \hat{\xi}^n = 0$ for $n \geq 2$

$$e^{\hat{\xi}\tau} = \left[\begin{array}{c|c} I & v\tau \\ \hline 0 & 1 \end{array}\right]$$

(b) if $\omega \neq \vec{0}$, things get complicated. in short,

$$e^{\hat{\xi}\tau} = \left[\begin{array}{c|c} e^{\hat{\omega}\tau} & (I - e^{\hat{\omega}\tau})\hat{\omega} v + \omega\omega^T v \tau \\ \hline 0 & 1 \end{array}\right]$$

what if we are given $g = \left[\begin{array}{c|c} R & p \\ \hline 0 & 1 \end{array}\right]$, ~~that~~ does there exist a ξ, τ pair such that

$$g = e^{\hat{\xi}\tau} \ ?$$

yes.

given R, $\omega = \ln R$ $\quad \tau = \|\omega\|$

$$p = (I - e^{\hat{\omega}\tau})\hat{\omega} v + \omega\omega^T \tau v$$

$$\Rightarrow v = \left((I - e^{\hat{\omega}\tau})\hat{\omega} + \omega\omega^T \tau\right)^{-1} p$$

↑ this is invertible.

$$\left[\cancel{I} - \left(\cancel{I} + \frac{\hat{\omega}}{\|\omega\|}\sin(\|\omega\|\tau) + \frac{\hat{\omega}^2}{\|\omega\|^2}\left(1-\cos(\|\omega\|\tau)\right)\right)\right]\frac{\hat{\omega} v}{\|\omega\|^2} + \frac{\omega\omega^T v}{\|\omega\|^2}\|\omega\|\tau$$

$$-\frac{\hat{\omega}^2}{\|\omega\|}\sin(\|\omega\|\tau)v - \frac{\hat{\omega}^3}{\|\omega\|^3}\left(1-\cos(\|\omega\|\tau)\right)v + \frac{\omega\omega^T v}{\|\omega\|^2}\|\omega\|\tau$$

$=$

$$+\frac{\|\omega\|^2\hat{\omega}}{\|\omega\|^3}\left(1-\cos(\|\omega\|\tau)\right)v - \frac{\hat{\omega}^2}{\|\omega\|^2}\sin(\|\omega\|\tau)v + \underbrace{\frac{\omega\omega^T}{\|\omega\|^2}v\|\omega\|\tau}_{\left(\frac{\hat{\omega}^2}{\|\omega\|^2}+I\right)v\|\omega\|\tau}$$

$=$

$$I + \frac{\hat{\omega}}{\|\omega\|}\left(1-\cos(\|\omega\|\tau)\right)v + \frac{\hat{\omega}^2}{\|\omega\|^2}\left(\sin(\|\omega\|\tau)v + v\|\omega\|\tau\right)$$

$=$

$$Iv + \frac{\hat{\omega}}{\|\omega\|}\left(1-\cos(\|\omega\|\tau)\right)v + \frac{\hat{\omega}^2}{\|\omega\|^2}\left(\|\omega\|\tau - \sin(\|\omega\|\tau)\right)v$$

$$= \left[I + \frac{\hat{\omega}}{\|\omega\|}\left(1-\cos(\|\omega\|\tau)\right) + \frac{\hat{\omega}^2}{\|\omega\|^2}\left(\|\omega\|\tau - \sin(\|\omega\|\tau)\right)\right]v$$

its actually

$$\left[I + \frac{\hat{\omega}}{\|\omega\|^2}\left(1-\cos(\|\omega\|\tau)\right) + \frac{\hat{\omega}^2}{\|\omega\|^2}\left(\|\omega\|\tau - \sin(\|\omega\|\tau)\right)\right]v$$

$2\cos\theta + 1 = \mathrm{trace}(R)$ $\quad \hat{\omega} = \frac{1}{2\sin(\theta)}(R-R^T)$

$\hat{\omega} = \ln R$

$$A^{-1} = I - \frac{1}{2}\hat{\omega} + \frac{2\sin(\|\omega\|) - \|\omega\|\left(1+\cos(\|\omega\|)\right)}{2\|\omega\|^2\sin\|\omega\|}\hat{\omega}^2 \qquad \omega \neq 0$$

if $\omega = 0$, $A = I$

What about the case SE(2) and ~~SE~~E(2)?

$\xi \in SE(2)$, $\xi = \begin{Bmatrix} v \\ \omega \end{Bmatrix}$ $\hat{\xi} = \left[\begin{array}{c|c} \hat{\omega} & v \\ \hline 0 & 0 \end{array}\right]$

$$e^{\hat{\xi}\tau} = \begin{cases} \text{if } \omega = 0, & e^{\hat{\xi}\tau} = \begin{bmatrix} 1 & v\tau \\ 0 & 1 \end{bmatrix} \\ \text{if } \omega \neq 0, & e^{\hat{\xi}\tau} = \left[\begin{array}{c|c} e^{\hat{\omega}\tau} & (I - e^{\hat{\omega}\tau})\hat{\omega} v \\ \hline 0 & 1 \end{array}\right] \end{cases}$$

$\xi = \ln g$?

$\omega = \ln R$

$p = (I - e^{\hat{\omega}\tau})\hat{\omega} v$

$\Rightarrow$

$v = \hat{\omega}^{-1}(I - e^{\hat{\omega}\tau})^{-1} p$

$v = -\frac{1}{\omega} J (I - e^{\hat{\omega}\tau})^{-1} p$

$\omega = g^3 = \theta$

if $\omega = 0$,

$v = \ \underline{\ \ }$