October 31 to Dec 2. (tentative) → 5 weeks

- 2 days for Thanksgiving
can make-up during
week of Dec. 5.

kinematic models relate joint motion ~~and~~ to end-effector motion

joint space

$$
\begin{tikzcd}
\text{Actuator Space} \arrow[r, bend right] & \text{Joint Space (Shape Space)} \arrow[l, bend right] \arrow[r, bend right, "\substack{\text{forward kinematics} \\ J^{B}_{body}(\theta)}"'] & \text{Configuration Space (Group Space)} \arrow[l, bend right, "\substack{J^{\#}_{body}(\theta) \\ \text{inverse kinematics}}"']
\end{tikzcd}
$$

most actuators controlled (low level) using forces/torques.

linear / revolute

DYNAMICS of robot manipulator describes how robot moves in response to actuator forces

- still ignoring the blue lines (this is more classical control)

control of servomechanisms & linear actuators.

→ allows to focus on mechanics of manipulators

obtain: nonlinear, second-order, ordinary differential equations

$$
\begin{tikzcd}
\text{Joint Space} \arrow[r, bend right, "\text{dynamics}"'] & \text{Configuration Space} \arrow[l, bend right, "\text{control}"']
\end{tikzcd}
$$

~~pos~~

joint-space control ⇐ control objective in joint space (shape space)

workspace control ← ~~g~~ configuration space (group space)

position control vs. force control

1) Dynamics
   - equations of motion
   - Lagrangian derivation

2) Control
   - position
   - force

trajectory control?

Advocate Lagrangian derivation

- systematic
- CRAIG CH6 devotes most of chapter to Newton approach, which is prone to errors
- minimizes # of variables
- large literature exists to analyze systems derived this way.

Typical Newton approach is to solve for system of n-particles/objects

$$m_i \ddot{z}_i = F_i \qquad z_i \in \mathbb{R}^3, \; i=1\ldots n \tag{1}$$

but our system is constrained, ~~eg~~ particles/objects have fixed relationship. These constraints encoded mathematically by

$$g_j(z_1, \ldots, z_n) = 0 \qquad j = 1 \ldots k$$

- called holonomic constraints
- for (1) to obey constraints, ~~we~~ must include "constraint forces"

~~eq~~

$$F_i = m_i \ddot{z}_i + \sum_{j=1}^{k} \Gamma_j \lambda_j$$

scale factor (affects magnitude of force).

constraint forces (vectors) (obtained from g_j)

c.

- λ_j - Lagrange multipliers.
  example: tilt-a-whirl, centrifugal acceleration.
- system has $3n+k$ equations & $3n+k$ unknowns

Now, note that

$$g_j(z_1,\dots,z_n)=0 \qquad j=1\dots k$$

provides k constraints

$\Rightarrow$

$m=3n-k$ degrees of freedom

$\Rightarrow$ there should exist $q_1\cdots q_m$ variables

and $f_1,\dots,f_n$ functions such that

$$z_i = f_i(q_1,\dots,q_m) \qquad i=1\dots n$$

so, $q_1,\dots,q_m$ & $z_1,\dots,z_n$

$g_j(z_1,\dots,z_n)=0$

are interchangeable

$q_1,\dots,q_m$ are called generalized coordinates.

Hamilton's variational principle says

Lagrangian: $L(q,\dot q) = T(q,\dot q) - V(q)$

Kinetic Energy　　potential energy

gen. coords.

Theorem (Lagrange's Equations)

The equations of motion for a mechanical w/ generalized coordinates $q \in \mathbb{R}^m$ and Lagrangian L are given by

$$\frac{d}{dt}\frac{\partial L}{\partial q} - \frac{\partial L}{\partial q} = \Upsilon_i \qquad i = 1, \ldots, m.$$

where Υ_i is the external force acting on the generalized coordinate.

- obtained from Hamilton's variational principle.

$$L(x,y,\dot x,\dot y) = \frac{1}{2}m\dot x^2 + \frac{1}{2}m\dot y^2 - mgy + \lambda(x^2+y^2-r^2)$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot q} - \frac{\partial L}{\partial q} = 0 \quad\Rightarrow\quad \boxed{\begin{aligned} m\ddot x - 2\lambda x &= 0 \\ m\ddot y + mg - 2\lambda y &= 0 \end{aligned}}$$

$$x^2+y^2-r^2=0 \quad\Rightarrow\quad 2x\dot x + 2y\dot y = 0$$

$$\boxed{2\dot x^2 + 2x\ddot x + 2\dot y^2 + 2y\ddot y = 0}$$

3 equs, 3 unknowns

$$\ddot x = \frac{2x}{m}\lambda$$

$$\ddot y = -\cancel{m}g + \frac{2y}{m}\lambda$$

$\Rightarrow$

$$\dot x^2 + x\left(\frac{2x}{m}\lambda\right) + \dot y^2 + y\left(-\cancel{m}g + \frac{2y}{m}\lambda\right) = 0$$

$\Rightarrow$

$$\frac{2}{m}(x^2+y^2)\lambda + \dot x^2 + \dot y^2 - \cancel{m}gy = 0$$

$\Rightarrow$

$$\lambda = \frac{m}{2(x^2+y^2)}\left(\cancel{m}gy - \dot x^2 - \dot y^2\right)$$

$\Rightarrow$

$$\ddot x = \frac{x}{r^2}\left(\cancel{m}gy - \dot x^2 - \dot y^2\right)$$

$$\ddot y = -\cancel{m}g + \frac{y}{r^2}\left(\cancel{m}gy - \dot x^2 - \dot y^2\right)$$

tension in rod: $\left\|\left(\frac{2x}{\cancel{m}}\lambda \quad \frac{2y}{\cancel{m}}\lambda\right)\right\| = |\lambda|\cdot 2r = \left|\frac{m}{r}(gy - \dot x^2 - \dot y^2)\right|$

• there is a systematic, but computationally demanding way to do this.

Alternative is to find coordinates that respect constraints

$$x^2+y^2-r^2=0$$

$x = r\sin\theta \qquad y = -r\cos\theta \qquad\qquad \theta=0 \Rightarrow$ hanging down

$\Rightarrow$

$$\dot{x} = r\cos\theta\,\dot{\theta} \qquad \dot{y} = r\sin\theta\,\dot{\theta}$$

$$L(\theta,\dot{\theta}) = \frac{1}{2}mr^2\dot{\theta}^2 + mgr\cos\theta$$

$$\frac{d}{dt}\frac{\partial L}{\partial\dot{\theta}} - \frac{\partial L}{\partial\theta} = mr^2\ddot{\theta} + mgr\sin\theta$$

$\Rightarrow$

$$\ddot{\theta} = -\frac{g}{r}\sin\theta$$

↑ simpler set of equations

- it's only in terms of $\theta, \dot{\theta}$.
- if we have n-joints, can we do that to our manipulator? YES!

Need to find $L(\theta,\dot{\theta}) = \frac{1}{2}\dot{\theta}^T M\dot{\theta} - V(\theta)$.

How do we do that?

The big question is, how does this relate to manipulators?

- they operate in Euclidean space (well, really special Euclidean space)
- described using joints in $\mathbb{R}^p \times \mathbb{T}^r$

\# revolute joints.

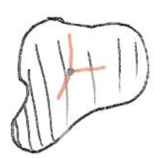
\# prismatic joints

Lagrangian can be well-defined in (special) Euclidean space ...

then transformed to joint space using same idea.

but it's a bit more complicated.

The Rigid Body Lagrangian (in SE(n)).



total mass: $m = \int_V \rho(x)\,dV$

center of mass: $x_c = \frac{1}{m}\int_V x\,\rho(x)\,dV$

object given by $g = (p, R)$

describes location/orientation of center of mass frame.

motion given by $\dot{g} = (\dot{p}, \dot{R})$

what is associated kinetic energy? - get kinetic energy of each "particle."

a particle at location x_c in body coordinates (center of mass)

moves w/velocity

$$\frac{d}{dt}(gx_c) = \dot{g}x_c = \dot{p} + \dot{R}x_c$$

$\Rightarrow$

$$T = \frac{1}{2}\int_V \rho(x_c)\, \|\dot{p} + \dot{R}x_c\|^2 dV$$

recall $\|x\|^2 = x^T x$

$$= \frac{1}{2}\int_V \rho(x_c)\left[\ \underbrace{\|\dot{p}\|^2}_{\frac{1}{2}m\|\dot{p}\|^2} + \underbrace{2\dot{p}^T\dot{R}x_c}_{\text{vanishes because body frame is at center of mass}} + \underbrace{\|\dot{R}x_c\|^2}_{\text{what is this?}}\ \right] dV$$

$\dot{R}$ is problematic. what's the alternative?

WORK THIS OUT TO GET: $\frac{1}{2}(\omega^b)^T \mathcal{I}\, \omega^b$

$$\frac{1}{2}\cdot 2\dot{p}^T\dot{R}\underbrace{\int_V x_c\,\rho(x_c)dV}_{\text{this is at origin of body frame.}}$$

$$= \frac{1}{2}m\|\dot{p}\|^2 + \frac{1}{2}\omega^T\mathcal{I}\omega$$

$$= \frac{1}{2}m\|v^b\|^2 + \frac{1}{2}\omega^T\mathcal{I}\omega$$

$$T = \frac{1}{2}(\xi^b)^T M\, \xi^b \qquad \text{where } M = \begin{bmatrix} mI & 0 \\ 0 & \mathcal{I} \end{bmatrix}$$

but what happened to T?

in generalized coords. $T(q, \dot{q})$

we did in $SE(n)$. $T(g, \dot{g})$ but ended up with $T(\xi)$, what's the deal?

if we have $T(q,\dot{q})$ we can find $\frac{\partial T}{\partial \dot{q}}, \frac{\partial T}{\partial q}$

but for $T(g,\dot{g})$, $\frac{\partial T}{\partial \dot{g}}, \frac{\partial T}{\partial g}$ may not make sense.

note that our derivation worked as follows:

$$T(g,\dot{g}) \to T(e, g^{-1}\dot{g}) \to T(e, \xi^b) \to T(\xi^b)$$

that's great because ξ^b is a vector, but what about $\frac{\partial T}{\partial \dot{g}}, \frac{\partial T}{\partial g}$?

it is possible to work out correct equations of motion using Hamilton's principle, but let's derive another way.

It's really the angle/orientation part that screws things up.

- let's look at that part only

Euler's Equations

angular momentum is angular inertia × angular velocity

consider spatial frame: $\mathcal{I}_s = Ad_R \mathcal{I}$

spatial ang. momentum: $\Pi = \mathcal{I}_s \omega^s$

d/dt gives equations of motion

$$\tau = \frac{d}{dt}(\mathcal{I}_s \omega^s) = \frac{d}{dt}(Ad_R \mathcal{I} \cdot \omega^s) = \frac{d}{dt}(Ad_R \mathcal{I}) \cdot \omega^s + Ad_R \mathcal{I} \cdot \dot{\omega}^s$$

$$\tau = \frac{d}{dt}(Ad_R I_\circ)\cdot\omega^s + Ad_R I\,\dot{\omega}^s$$

$$\downarrow$$

???

$$\downarrow$$

$$\begin{aligned}
\frac{d}{dt}(Ad_R I)\cdot\omega^s &= \frac{d}{dt}(R I R^{-1})\,\omega^s \\
&= \frac{d}{dt}(R I R^T)\,\omega^s \\
&= (\dot{R} I R^T + R I \dot{R}^T)\,\omega^s \\
&= \dot{R}R^T(R I R^T)\,\omega^s + (R I R^T) R \dot{R}^T \omega^s \\
&= \hat{\omega}^s I_s \omega^s + I_s (\dot{R}R^T)^T \omega^s \\
&= \hat{\omega}^s I_s \omega^s + I_s (\hat{\omega}^s)^T \omega^s \\
&= \omega^s \times I_s \omega^s - I_s\cancel{(\omega^s \times \omega^s)} \\
&= \omega^s \times I_s \omega^s
\end{aligned}$$

$\Rightarrow$

$$\tau = I_s \dot{\omega}^s + \omega^s \times I_s \omega^s \qquad \text{(Euler's Equations)}$$

in spatial coords.

~~aside: consider $R(t)$ where $R(0) = I$~~
~~aligned w/ spatial frame~~

~~$\left.\frac{d}{dt}\right|_{t=0} Ad_{R(t)}\alpha =$~~

constant value since in body coords.

can do same in body coords.: $\tau^b = I\dot{\omega}^b + \omega^b \times I\omega^b$

in body coords. $\tau^b = R^T\tau = R^{-1}\tau$

Newton-Euler Equations

What about if we reintroduce the position part?

consider linear momentum $m\dot{p}$

$\Rightarrow$

$$\frac{d}{dt}(m\dot{p}) = \frac{d}{dt}(mRv^b) = mR\dot{v}^b + m\dot{R}v^b = f$$

$\Rightarrow$

$$mR\dot{v}^b + m\dot{R}v^b = f$$

$\Rightarrow$ R^{-}

$$m\dot{v}^b + mR^{-1}\dot{R}v^b = R^{-1}f$$

$$m\dot{v}^b + m\hat{\omega}^b v^b = f^b$$

$\Rightarrow$

$$m\dot{v}^b + m\omega^b \times v^b = f^b$$

$\Longrightarrow$ combine w/angular part:

$$\begin{bmatrix} mI & 0 \\ 0 & I \end{bmatrix} \begin{Bmatrix} \dot{v}^b \\ \dot{\omega}^b \end{Bmatrix} + \begin{bmatrix} \omega^b \times mv^b \\ \omega^b \times I\omega^b \end{bmatrix} = \begin{Bmatrix} f^b \\ \tau^b \end{Bmatrix}$$

(Newton-Euler equations)

in body coordinates.

$F^b = \begin{Bmatrix} f^b \\ \tau^b \end{Bmatrix}$ is called a wrench.