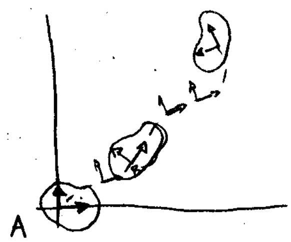
Velocities in SE(2).

So far we have dealt w/ displacements (transformations) and not w/ general rigid body motions.

What if we now thought of the transformation as arising from rigid body motion for a fixed time?

$g(t) : [t_0, t_1] \rightarrow G$ What is the velocity of the body?



in static frame A, it is $\dot{g}(t)$.

what is $\dot{g}(t)$? depends on representation

if $g = (\vec{d}(t), \theta(t))$, then

$$\dot{g} = (\dot{\vec{d}}(t), \dot{\theta}(t))$$

what about if $g = \begin{bmatrix} R(\theta(t)) & d(t) \\ 0 & 1 \end{bmatrix} \Rightarrow \dot{g}(t) = \begin{bmatrix} \frac{d}{dt}(R(\theta(t))) & \dot{d}(t) \\ 0 & 1 \end{bmatrix}$

but $R(t) = \begin{bmatrix} \cos(\theta(t)) & -\sin(\theta(t)) \\ \sin(\theta(t)) & \cos(\theta(t)) \end{bmatrix}$

$\Rightarrow$

$$\frac{d}{dt} R(\theta(t)) = \frac{dR}{d\theta} \cdot \frac{d\theta}{dt} = \begin{bmatrix} -\sin(\theta(t)) & -\cos(\theta(t)) \\ \cos(\theta(t)) & -\sin(\theta(t)) \end{bmatrix} \dot{\theta}(t) = DR \cdot \dot{\theta}(t)$$

$\dot{g}(t) = \begin{bmatrix} DR(\theta(t)) \cdot \dot{\theta}(t) & \dot{d}(t) \\ 0 & 0 \end{bmatrix}$ ← can describe this differently, want a more intuitive description.

consider, $e(t)\,\dot g(t)$ - clever form of identity.

$$\dot g(t) = \overbrace{g(t)\, g^{-1}(t)}\,\dot g(t) = g(t)\cdot \begin{bmatrix} R^{-1}(\theta(t)) & -R^{-1}d(t) \\ 0 & 1 \end{bmatrix}\begin{bmatrix} DR(\theta(t))\,\dot\theta(t) & \dot{\vec d}(t) \\ 0 & 0 \end{bmatrix}$$

$$= g(t)\cdot \begin{bmatrix} R^{-1}(\theta(t))\,DR(\theta(t))\,\dot\theta(t) & R^{-1}(\theta(t))\,\dot{\vec d}(t) \\ 0 & 0 \end{bmatrix}$$

aside:

$$R^{-1}(\theta(t))\cdot DR(\theta(t))\,\dot\theta(t)$$

$$= \begin{bmatrix} \cos(\theta(t)) & \sin(\theta(t)) \\ -\sin(\theta(t)) & \cos(\theta(t)) \end{bmatrix}\begin{bmatrix} -\sin(\theta(t)) & -\cos(\theta(t)) \\ \cos(\theta(t)) & -\sin(\theta(t)) \end{bmatrix}\dot\theta(t)$$

$$= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\dot\theta(t)$$

$$= g(t)\cdot \begin{bmatrix} \hat{\dot\theta}(t) & R^{-1}(\theta(t))\,\dot{\vec d}(t) \\ 0 & 1 \end{bmatrix} = g(t)\, v^b(t)$$

this is like v^B versus v^A.

called the body velocity

can also consider identity on other side

what is this?

$$\dot g(t) = \dot g(t)\, g^{-1}(t)\, g(t) = \begin{bmatrix} DR(\theta(t))\,R^{-1}(\theta(t))\,\dot\theta & -DR(\theta(t))\,\dot\theta(t)\cdot R^{-1}(\theta(t))\,d(t) + \dot{\vec d}(t) \\ 0 & 0 \end{bmatrix} g(t)$$

$$\begin{bmatrix} -s & -c \\ c & -s \end{bmatrix}\begin{bmatrix} c & s \\ -s & c \end{bmatrix} \qquad \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

spatial velocity

$$= \begin{bmatrix} \hat{\dot\theta}(t) & -\hat{\dot\theta}(t)\,\vec d(t) + \dot{\vec d}(t) \\ 0 & 1 \end{bmatrix}$$

$\cancel{-DR(\theta(t))\dot\theta(t)}$

how do we interpret these velocities?

- body is pretty intuitive
- spatial is a bit more difficult to grasp.

rotation of coordinates

$$\hat{v}^s = \dot{g}(t)\, g^{-1}(t) = \left[\begin{array}{c|c} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \dot{\theta}(t) & -\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} d(t) \cdot \dot{\theta}(t) + \dot{d}(t) \\ \hline 0 & 1 \end{array}\right]$$

velocity of a (possibly imaginary) point on the rigid body which is travelling through the origin of the spatial frame at time t.

- If spatial velocity is velocity of point on rigid body w/ a different frame, what relationship does it have to spatial velocity? $\longrightarrow$ Adjoint!

  it's really velocity from a different reference point.

Note that

$$\dot{g}(t) = g(t)\, \hat{v}^b(t) = \hat{v}^s(t)\, g(t)$$

$$\Rightarrow \quad \hat{v}^s(t)\, g(t) = g(t)\, \hat{v}^b(t)$$

$$\Rightarrow \; (\;) * g^{-1}(t) \quad \hat{v}^s(t) = g(t)\, \hat{v}^b(t)\, g^{-1}(t)$$

$$\Rightarrow \quad v^s(t) = Ad_{g(t)}\, v^b(t)$$

Can we show this in homogeneous coordinates?

$$v^s(t) = \left[\begin{array}{c|c} R & d \\ \hline 0 & 1 \end{array}\right] \left[\begin{array}{c|c} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\dot{\theta} & R^{-1}\dot{d} \\ \hline 0 & 0 \end{array}\right] \left[\begin{array}{c|c} R^{-1} & -R^{-1}d \\ \hline 0 & 1 \end{array}\right]$$

$$= \left[\begin{array}{c|c} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} R^{-1}\dot{\theta} & -\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} R^{-1} d\dot{\theta} + R^{-1}\dot{d} \\ \hline 0 & 0 \end{array}\right]$$

$$= \left[\begin{array}{c|c} R\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} R^{-1}\dot{\theta} & -R\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} R^{-1} d\dot{\theta} + RR^{-1}\dot{d} \\ \hline 0 & 0 \end{array}\right]$$

$$= \left[\begin{array}{c|c} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\dot{\theta} & -\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} d\dot{\theta} + \dot{d} \\ \hline 0 & 0 \end{array}\right]$$

$\rightarrow$ so, it checks out ~~not~~ in homogeneous coordinates, as it well should!

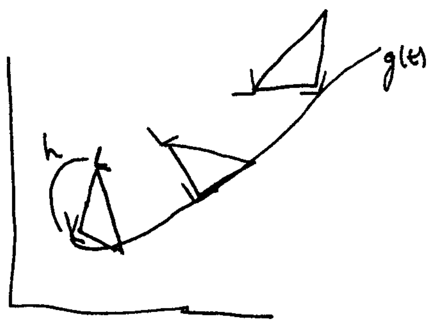
and we see that spatial velocity is just a change of reference frame of the body velocity.

To simplify notation in the future, we will consider the following shorthand:

$$\hat{\omega} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\omega = \begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix}$$

so, $\hat{\dot{\theta}} = \begin{bmatrix} 0 & -\dot{\theta} \\ \dot{\theta} & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\dot{\theta}$

what if we are dealing w/a rigid body & different points on it.



$$g_1(t) = g(t)$$

$$g_2(t) = g(t)h$$

$$\dot{g}_1(t) = \dot{g}(t)$$

$$\dot{g}_2(t) = \dot{g}(t)h = \begin{bmatrix} DR\cdot\dot{\theta} & \dot{\vec{d}} \\ 0 & 0 \end{bmatrix}\begin{bmatrix} R_h & \vec{d}_h \\ 0 & 1 \end{bmatrix}$$

$$= \left[\begin{array}{c|c} (DR\cdot\dot{\theta})R_h & DR\dot{\theta}\,\vec{d}_h + \dot{\vec{d}} \\ \hline 0 & \mathbf{0} \end{array}\right]$$

$$\dot{g}_1(t) = g(t)\left(g^{-1}(t)\dot{g}(t)\right) = g(t)\begin{bmatrix} \hat{\dot{\theta}}(t) & R^{-1}\dot{\vec{d}}(t) \\ 0 & \mathbf{0} \end{bmatrix} = g(t)\,v^b(t)$$

$$= \begin{bmatrix} \hat{\dot{\theta}}(t) & -\hat{\dot{\theta}}\,\vec{d}(t) + \dot{\vec{d}}(t) \\ 0 & \mathbf{0} \end{bmatrix} g(t) = v^s(t)\,g(t)$$

$$\dot{g}_2(t) = g(t)h\left(h^{-1}g^{-1}(t)\dot{g}(t)h\right) = g(t)h\begin{bmatrix} R_h^{-1} & -R_h^{-1}d_h \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \hat{\dot{\theta}} & R^{-1}\dot{d}(t) \\ 0 & 0 \end{bmatrix}h$$

$$= g(t)h\;Ad_{h^{-1}}\left(g^{-1}(t)\dot{g}(t)\right) = g_2(t)\;Ad_{h^{-1}}v^b(t)$$

$$\Rightarrow \quad v_2^b(t) = Ad_{h^{-1}}\,v^b(t)$$

What does adjoint of a ~~velo~~ spatial/ body velocity look like?

well, let $\quad v = (v, \omega) = (v, \hat{\omega}) = \left[\begin{array}{c|c} \hat{\omega} & v \\ \hline 0 & 0 \end{array}\right]$

$$\mathrm{Ad}_g v = \left[\begin{array}{c|c} R & p \\ \hline 0 & 1 \end{array}\right]\left[\begin{array}{c|c} \hat{\omega} & v \\ \hline 0 & 0 \end{array}\right]\left[\begin{array}{c|c} R^{-1} & -R^{-1}p \\ \hline 0 & 1 \end{array}\right]$$

$$= \left[\begin{array}{c|c} R & p \\ \hline 0 & 1 \end{array}\right]\left[\begin{array}{c|c} \hat{\omega}R^{-1} & -\omega R^{-1}p + v \\ \hline 0 & 0 \end{array}\right]$$

$$= \left[\begin{array}{c|c} R\hat{\omega}R^{-1} & -R\omega R^{-1}p + Rv \\ \hline 0 & 0 \end{array}\right]$$

special SE(2) property.

$$= \left[\begin{array}{c|c} \hat{\omega} & \hat{\omega}p + Rv \\ \hline 0 & 0 \end{array}\right]$$

$$= (\hat{\omega}, \hat{\omega}p + Rv) \qquad \hat{\omega}p = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} p \cdot \omega$$

$$= \begin{bmatrix} 0 & -p_y \\ p_x & 0 \end{bmatrix} \omega$$

$$= Jp, \quad J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$\Rightarrow$

$$\mathrm{Ad}_g \begin{Bmatrix} v \\ \omega \end{Bmatrix} = \left[\begin{array}{c|c} R & Jp \\ \hline 0 & 1 \end{array}\right] \begin{Bmatrix} v \\ \omega \end{Bmatrix}$$

only holds for SE(2).

Properties of these Special Velocities:

why do I care about having velocity in form:

$$\begin{Bmatrix} V \\ \omega \end{Bmatrix} \approx \left[\begin{array}{c|c} \begin{bmatrix}0&-1\\1&0\end{bmatrix}\omega & V \\ \hline 0 & 0\end{array}\right] = \left[\begin{array}{c|c}\hat{\omega} & V\\ \hline 0&0\end{array}\right]$$

well, this is because $\begin{bmatrix}0&-1\\1&0\end{bmatrix}\omega$ is a special type of matrix: consider

$$R \in SO(2) \Rightarrow R^T R = I$$

$$\Rightarrow \frac{d}{dt} R^T R = 0$$

$$\dot{R}^T R + R^T \dot{R} = 0$$

$$(R^T \dot{R})^T + R^T \dot{R} = 0$$

$$R^T \dot{R} = -(R^T \dot{R})^T$$

~~Also~~ what is this?

if $\hat{\omega} = R^T \dot{R}$, then

$$\hat{\omega} = -\hat{\omega}^T$$

That means $\hat{\omega}$ is skew-symmetric!

sums & differences of skew-symmetric matrices are still skew-symmetric!

We have

a) ability to add & subtract v's. (vectors)

b) ability to add & subtract $\hat{\omega}$'s.

$\Rightarrow$

this is an element that can be added and subtracted

$$\left[\begin{array}{c|c} \hat{\omega} & v \\ \hline 0 & 0 \end{array}\right] = \left[\begin{array}{c|c} \hat{\omega}_1 & v_1 \\ \hline 0 & 0 \end{array}\right] + \left[\begin{array}{c|c} \hat{\omega}_2 & v_2 \\ \hline 0 & 0 \end{array}\right]$$

this is legitimate!

$$= \left[\begin{array}{c|c} \hat{\omega}_1 + \hat{\omega}_2 & v_1 + v_2 \\ \hline 0 & 0 \end{array}\right]$$

the space of skew-symmetric 2×2 matrices is called $\mathfrak{so}(2)$. its operation is addition.

note that ~~vectors of a~~ velocities of $E(2)$ are still in $E(2)$.

$\Rightarrow$

$$\underline{\xi} = \begin{Bmatrix} v \\ \omega \end{Bmatrix} \in \cancel{\mathfrak{se}(2)}\ E(2) \times \mathfrak{so}(2).$$

Thus we have this special space

$$\mathfrak{se}(2) = E(2) \times \mathfrak{so}(2)$$

for describing vectors. It is called the Lie algebra of $SE(2)$.

Define Lie algebra? $[\xi_1, \xi_2]$

Properties of $SE(3)$, $\mathfrak{se}(3)$.

Action of $SE(3)$ on $\mathfrak{se}(3)$:

$$g \cdot \xi = \left[\begin{array}{c|c} R & p \\ \hline 0 & 1 \end{array}\right]\left[\begin{array}{c|c} \hat{\omega} & v \\ \hline 0 & 0 \end{array}\right]$$

$$= \left[\begin{array}{c|c} R\hat{\omega} & Rv \\ \hline 0 & 0 \end{array}\right]$$

Adjoint action:

$$g\xi g^{-1} = \left[\begin{array}{c|c} R & p \\ \hline 0 & 1 \end{array}\right]\left[\begin{array}{c|c} \hat{\omega} & v \\ \hline 0 & 0 \end{array}\right]\left[\begin{array}{c|c} R^{-1} & -R^{-1}p \\ \hline 0 & 1 \end{array}\right]$$

$$= \begin{pmatrix} R\hat{\omega} & Rv \\ 0 & 0 \end{pmatrix}\left[\begin{array}{c|c} R^{-1} & -R^{-1}p \\ \hline 0 & 1 \end{array}\right]$$

$$= \left[\begin{array}{c|c} R\hat{\omega}R^{-1} & -R\hat{\omega}R^{-1}p + Rv \\ \hline 0 & 0 \end{array}\right]$$

what is $R\hat{\omega}R^{-1}$? $\quad R\hat{\omega}R^{-1} = (R\omega)^{\wedge}$

$\widehat{(R\omega)}$ $\qquad \hat{a}b = -\hat{b}a$

$$= \left[\begin{array}{c|c} \widehat{R\omega} & -\widehat{(R\omega)}p + Rv \\ \hline 0 & 0 \end{array}\right]$$

so $\mathrm{Ad}_g \xi = \mathrm{Ad}_g \begin{Bmatrix} v \\ \omega \end{Bmatrix} = \begin{bmatrix} R & \hat{p}R \\ 0 & R \end{bmatrix}$