Aug 22 to Sep 23 → 1 month | 5 weeks

1) Transformations in SE(2)
   - change of bases.
   - analogy w/ Euclidean space

2) Transformations of Euclidean space
   - SE(2), SE(3) - Adjoint
   - Lie groups ; homogeneous notation
   - analogy w/ Euclidean space
   - forward kinematics (intro)

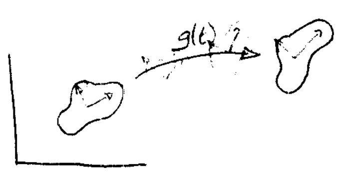
3) Velocities & Lie algebras
   - vectors of Lie groups 1) as curve velocities 2) as displacements
   - Lie algebras
   - exp and log function
   - Adjoint of Lie algebra elements
   - Manipulator Jacobian (intro)
   - Product of Exponentials for Kinematic Chains.

ECE 4560 Craig 2.1-2.6

planar kinematics
- definitions
- transformations & coordinates
- reference frames

**Definition.** A <u>rigid body</u> is a collection of points/particles which have a fixed relationship amongst themselves.

**Definition.** A <u>rigid body motion</u> describes how the individual particles of a rigid body move as a function of time. Equivalently, a rigid body motion is the motion of the body fixed reference frame.




$g(t)$ ← describes the rigid body motion

**Definition.** A <u>displacement/transformation</u> is the movement or motion of a rigid body w/out reference to time-scale.

- from $g(0)$ to $g(T)$ is displacement

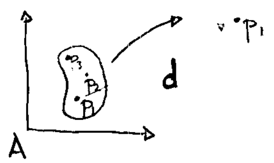
**Definition.** A <u>planar rigid body motion</u> is rigid body motion during which all of the particles remain in a plane (or a set of parallel planes)

**Definition.** The degrees of freedom (DOF) of a motion is the minimal number of independent variables needed to uniquely specify the motion of an object.

How many degrees of freedom in planar motion?

**Proposition.** Planar motion has 3 degrees of freedom.

proof




1) location of particle $p_i$ denoted by $(x_i, y_i)$

2) consider $p_1$. it has 2 degrees of freedom since $d = (d_x, d_y)$ is arbitrary.

3) consider $p_2$. it must satisfy some fixed relationship w/respect to $p_1$, by definition of rigid body.

$\Rightarrow$

i.e., fixed distance $(x_2 - x_1)^2 + (y_2 - y_1)^2 = d_{12}^2$


possible locations of $p_2$

2 DOF − 1 constraint = 1 DOF

4) consider $p_3$. it has 2DOF − 2 constraints = 0 DOF


only 2 options $\Rightarrow$ 0 DOF.

5) consider $p_i$, $i > 3$. 2DOF, $i-1$ constraints

$\Rightarrow$ 2 real constraints, $i-3$ redundant ones

2DOF − 2 constraints $\Rightarrow$ 0 DOF.

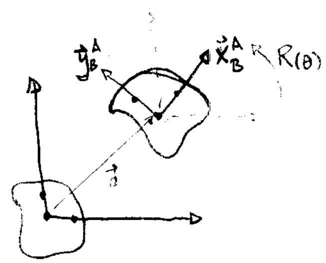
$\Rightarrow$

3 total degrees of freedom

in general, will have (for n particles) $2n$ DOF $-$ $(2(n-2)+1)$ constraints $=$ 3 DOF

COORDINATES

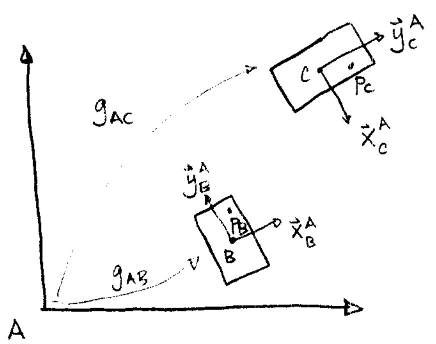
The 3DOF of freedom are planar position + orientation




configuration of rigid body : $g = (x, y, \theta)$

or equivalently $(x, y, R(\theta))$ $(\vec{d}, R(\theta))$ $\Bigg\}$ $R(\theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$

In a sense the coordinates also describe a displacement/transformation
- in above example, from origin to new location
- let's consider a more complicated example




to see this consider the point p on the rigid body

- in coordinate frame of object, point p is at : $p = \alpha_1 \vec{x} + \alpha_2 \vec{y} = (\alpha_1, \alpha_2)$

e.g. $p_B^B = \alpha_1 \vec{x}_B^A + \alpha_2 \vec{y}_B^A$
$p_C^C = \alpha_1 \vec{x}_C^A + \alpha_2 \vec{y}_C^A$

where is the point $p_C^B$ located in B's reference system?

well, $\vec{P}_c^B = \vec{d}_{Bc} + \vec{P}'$

$\uparrow$ due to rotation of axes.

but, we know that it is located at :

$$\vec{P}_c = \alpha_1 \vec{x}_c^A + \alpha_2 \vec{y}_c^A = (\alpha_1, \alpha_2)^A$$

just have to convert to B frame

$$\vec{P}_c = \alpha_1 \left( \vec{x}_B^A \cos(\theta_{Bc}) - \vec{y}_B^A \sin(\theta_{Bc}) \right) + \alpha_2 \left( \vec{x}_B^A \sin(\theta_{Bc}) + \vec{y}_B^A \cos(\theta_{BC}) \right)$$

$$= \left(\alpha_1 \cos(\theta_{Bc}) - \alpha_2 \sin(\theta_{Bc})\right) \vec{x}_B^A + \left(\alpha_1 \sin(\theta_{Bc}) + \alpha_2 \cos(\theta_{Bc})\right) \vec{y}_B^A = R(\theta_{Bc}) \begin{Bmatrix} \alpha_1 \\ \alpha_2 \end{Bmatrix}^A = R(\theta_{Bc})\vec{P}$$

$\Rightarrow$

$$\vec{P}_c^B = \vec{d}_{BC} + R(\theta_{Bc})\vec{P}$$

where $R(\theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$

so, the point $p_B$ moves to $p_c$ according to $(\vec{d}_{BC}, R(\theta_{Bc}))$

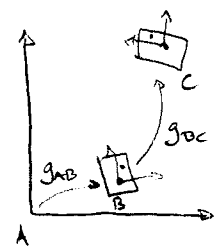
thus we see that $(\vec{d}, R(\theta))$ can be used to describe either a <u>configuration</u> or a <u>transformation</u>.

if $\vec{d}_{Bc} = 0 \Rightarrow$ pure rotation

if $\theta_{Bc} = 0$ or $R(\theta_{AB}) = \mathbb{1} \Rightarrow$ pure translation.

Is it possible to represent multiple displacements via some product?

- Consider two displacements




$P_C$ in frame B is $\quad P_C^B = \vec{d}_{BC} + R(\theta_{BC})\vec{p}$

$P_B$ in frame A is $\quad P_B^A = \vec{d}_{AB} + R(\theta_{AB})\vec{p}$

$\Rightarrow$

$P_C$ in frame A is: $\quad P_C = \vec{d}_{AB} + R(\theta_{AB})\vec{P}_C^B$

$$= \vec{d}_{AB} + R(\theta_{AB})(\vec{d}_{BC} + R(\theta_{BC})\vec{p})$$

$$= \vec{d}_{AB} + R(\theta_{AB})\vec{d}_{BC} + R(\theta_{AB})R(\theta_{BC})\vec{p}$$

$\Rightarrow$

$$g_{AC} = g_{AB} \cdot g_{BC} = (d, R) = (\vec{d}_{AB} + R(\theta_{AB})\vec{d}_{BC},\ R(\theta_{AB})R(\theta_{BC}))$$

is there an identity transformation, e.g. one that does nothing?

$(\vec{d}, R)\vec{p} = \vec{p} \quad \Rightarrow \quad \vec{d} + R\vec{p} = \vec{p}$ well $\vec{d} = 0$, $R = I$ does the trick

$\Rightarrow$

$$e = (\vec{0}, \mathbb{1})$$

is there an inverse transformation?

$$(\vec{d}, R) \cdot (\vec{d}_i, R_i) = (\vec{0}, \mathbb{1})$$

$\Rightarrow$

$$(\vec{d} + R\vec{d}_i,\ R R_i) = (\vec{0}, \mathbb{1})$$

$\Rightarrow$

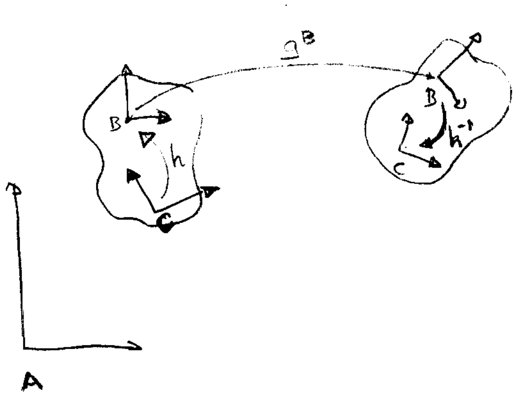
$$\vec{d} + R\vec{d}_i = 0, \quad R R_i = \mathbb{1}$$

$\Rightarrow$

$$\vec{d}_i = -R^{-1}\vec{d}, \quad R_i = R^{-1}$$

$$\Rightarrow \begin{array}{c} g^{-1} \\ \parallel \\ (\vec{d}, R)^{-1} \end{array} = (-R^{-1}\vec{d}, R^{-1})$$

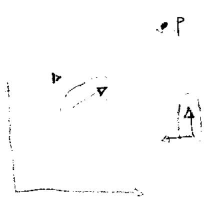
Can use these operations to understand how to change reference frame of a displacement




$$g^C = ? = h g^B h^{-1}$$

called adjoint operation.

idea behind pole.




Theorem. Every planar displacement is equivalent to rotation about a point.

this point is called the pole.

$$g \cdot p = p$$

$$\Rightarrow$$

$$(d, R) \cdot p = p$$

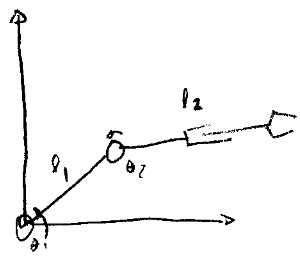
$$\Rightarrow$$

$$d + Rp = p$$

This point is called the pole

Here it is considered in reference frame B.

Can apply these ideas to manipulators:




what is end-effector's configuration?

$$g_e = \begin{Bmatrix} d_1 \cos(\theta_1) + d_2 \cos(\theta_1+\theta_2) \\ d_1 \sin(\theta_1) + d_2 \sin(\theta_1+\theta_2) \\ \theta_1+\theta_2 \end{Bmatrix}$$

as a product, we consider changes from one joint to another.

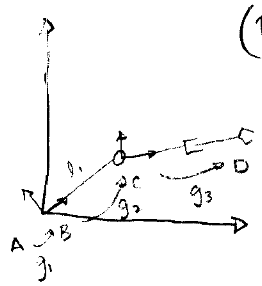
$$g_e = g_1 g_2 g_3$$

$$= (\vec{0}, R(\theta_1)) \cdot (\vec{d}_1, R(\theta_2)) \cdot (\vec{d}_2, I)$$

$$= (\vec{0}, R(\theta_1)) \cdot (\vec{d}_1 + R(\theta_2)\vec{d}_2, R(\theta_2))$$

$\downarrow$ verification

$$= (R(\theta_1)\vec{d}_1 + R(\theta_1)R(\theta_2)\vec{d}_2, R(\theta_1)R(\theta_2))$$




$$= (R(\theta_1)\vec{d}_1 + R(\theta_1+\theta_2)\vec{d}_2, R(\theta_1+\theta_2))$$

The space of planar rigid body configurations/transformations is called SE(2).

SE - special Euclidean.

It is a Lie group.

**Definition.** A group is a mathematical structure consisting of a set of elements that can be indexed; set can be finite, infinite, or continuous. Let $G = \{g_j\}$ $j \in I$: the index set, denote this set of elements. The group has:

1) an associative group operation, denoted by $*$, called the group product.
2) there is a unique element, $e$, called the identity element such that

$$e * g_j = g_j \qquad \forall g_j \in G$$

3) for every $g_j \in G$, there exists an inverse element $g_j^{-1}$, called the inverse, such that

$$g_j * g_j^{-1} = e$$

**Definition.** A Lie group is a group $G$ which is also a smooth manifold and for which the group product and inverse are smooth.

SE(2) is a Lie group.

↳ note that we saw multiple representations for SE(2), want to consider a special version, called homogeneous representation.

# Homogeneous Coordinates

$$(\vec{d}, R) \longrightarrow \left[\begin{array}{c|c} R & \vec{d} \\ \hline 0 & 1 \end{array}\right] \text{ matrix}$$

$R$ - $2\times 2$

$d$ - $2\times 1$

$0$ - $1\times 2$

$1$ - $1\times 1$

$$\left[\begin{array}{cc|c} x & x & x \\ x & x & x \\ \hline x & x & 1 \end{array}\right]$$

this is invertible, and products work out correctly as does identity.

$$e = \begin{bmatrix} I & \vec{0} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$g_1 g_2 = \begin{bmatrix} R_1 & \vec{d}_1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} R_2 & \vec{d}_2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} R_1R_2 & R_1\vec{d}_2 + \vec{d}_1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} R_1R_2 & \vec{d}_1 + R_1\vec{d}_2 \\ 0 & 1 \end{bmatrix}$$

$$\text{vs} \quad (\vec{d}_1, R_1)\cdot(\vec{d}_2, R_2) = (\vec{d}_1 + R_1\vec{d}_2, R_1R_2)$$

Homework: $g \cdot g_i = e$ ? what is $g_i$ ?

what about points?

now represented by $\begin{bmatrix} \vec{d} \\ 1 \end{bmatrix} = \begin{Bmatrix} x \\ y \\ 1 \end{Bmatrix}$

and vectors?

$$\begin{bmatrix} \vec{v} \\ 0 \end{bmatrix} = \begin{Bmatrix} v_x \\ v_y \\ 0 \end{Bmatrix}$$

recall $V = P_1 - P_2$

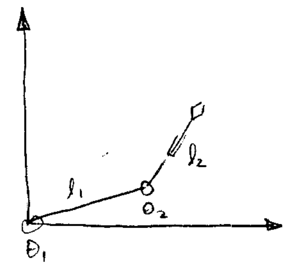
$$\Rightarrow \begin{Bmatrix} v_x \\ v_y \\ 0 \end{Bmatrix} = \begin{Bmatrix} x_1 \\ y_1 \\ 1 \end{Bmatrix} - \begin{Bmatrix} x_2 \\ y_2 \\ 1 \end{Bmatrix} = \begin{Bmatrix} x_1 - x_2 \\ y_1 - y_2 \\ 0 \end{Bmatrix}$$

$g \cdot p \to$ gives a point

$g \cdot v \to$ gives a vector.

# WORKING WITH SE(2)

Today, will consider a manipulator task using our example manipulator:




recall configuration of the end effector

$$g_e = g_1 g_2 g_3 = \left[\begin{array}{c|c} R(\theta_1) & 0 \\ \hline 0 & 1 \end{array}\right] \left[\begin{array}{c|c} R(\theta_2) & \vec{d}_1 \\ \hline 0 & 1 \end{array}\right] \left[\begin{array}{c|c} I & \vec{d}_2 \\ \hline 0 & 1 \end{array}\right]$$

$$= \left[\begin{array}{c|c} R(\theta_1+\theta_2) & R(\theta_1)\vec{d}_1 + R(\theta_1+\theta_2)\vec{d}_2 \\ \hline 0 & 1 \end{array}\right]$$

The manipulator is designed such that $l_1 = 1$, $l_2 \in [\frac{1}{2}, 2]$

$\theta_1 \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, $\theta_2 \in [-\frac{3\pi}{4}, \frac{3\pi}{4}]$

$\Rightarrow$ $\vec{d}_1 = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix}$ $\vec{d}_2 = \begin{Bmatrix} l_2 \\ 0 \end{Bmatrix}$

what is the end effector configuration for: $\theta_1 = \frac{\pi}{6}$, $\theta_2 = \frac{\pi}{6}$, $l_2 = 1$ ?

$$g_e = \left[\begin{array}{c|c} R(\frac{\pi}{3}) & R(\frac{\pi}{6})\vec{d}_1 + R(\frac{\pi}{3})\vec{d}_2 \\ \hline 0 & 1 \end{array}\right]$$

$$R(\tfrac{\pi}{6})\begin{Bmatrix} 1 \\ 0 \end{Bmatrix} + R(\tfrac{\pi}{3})\begin{Bmatrix} 1 \\ 0 \end{Bmatrix} = \begin{bmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{bmatrix}\begin{Bmatrix} 1 \\ 0 \end{Bmatrix} + \begin{bmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{bmatrix}\begin{Bmatrix} 1 \\ 0 \end{Bmatrix}$$

$$= \begin{Bmatrix} (1+\sqrt{3})/2 \\ (1+\sqrt{3})/2 \end{Bmatrix}$$

$\Rightarrow$

$$g_e = \left[\begin{array}{c|c} R(\frac{\pi}{3}) & \begin{matrix} (1+\sqrt{3})/2 \\ (1+\sqrt{3})/2 \end{matrix} \\ \hline 0 & 1 \end{array}\right] \cong \underline{\left((1+\sqrt{3})/2,\ (1+\sqrt{3})/2,\ \pi/3\right)}$$

what about for $\theta_1 = \frac{\pi}{6}$, $\theta_2 = -\frac{\pi}{6}$, $l_2 = 2$

$$g_e = \left[\begin{array}{c|c} R(0) & R(\frac{\pi}{6})\vec{d}_1 + R(0)\vec{d}_2 \\ \hline 0 & 1 \end{array}\right]$$

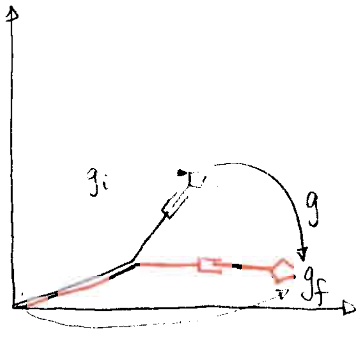
$$R(\pi/6)\vec{d}_1 + R(0)\vec{d}_2 = \begin{bmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{bmatrix} \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} 2 \\ 0 \end{Bmatrix}$$

$$= \begin{Bmatrix} 2 + \sqrt{3}/2 \\ 1/2 \end{Bmatrix}$$

$\Rightarrow$

$$g_e = \left[\begin{array}{c|c} R(0) & \begin{matrix} 2+\sqrt{3}/2 \\ 1/2 \end{matrix} \\ \hline 0 & 1 \end{array}\right] \cong \underline{(2 + \sqrt{3}/2,\ 1/2,\ 0)}$$

what is the transformation that the end effector undergoes:




according to diagram,

$$\underbrace{g_f}_{\text{final configuration}} = \underbrace{g_i}_{\text{initial configuration}} \overbrace{g}^{\text{transformation}}$$

we can solve for $g$:

$$g = g_i^{-1} g_f = \left[\begin{array}{c|c} R(\pi/3) & \begin{matrix} (1+\sqrt{3})/2 \\ (1+\sqrt{3})/2 \end{matrix} \\ \hline 0 & 1 \end{array}\right]^{-1} \left[\begin{array}{c|c} R(0) & \begin{matrix} 2+\sqrt{3}/2 \\ 1/2 \end{matrix} \\ \hline 0 & 1 \end{array}\right]$$

- this is like part 1b of your homework!

$\Rightarrow$

$$g = \left[\begin{array}{c|c} R(-\pi/3) & -R(-\pi/3)\begin{Bmatrix}(1+\sqrt{3})/2\\ (1+\sqrt{3})/2\end{Bmatrix} \\ \hline 0 & 1\end{array}\right]\left[\begin{array}{c|c} R(0) & \begin{matrix}(2+\sqrt{3})/2\\ 1/2\end{matrix} \\ \hline 0 & 1\end{array}\right]$$

can combine since same rotation matrix.

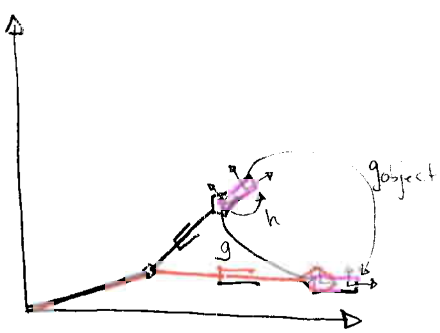
$$= \left[\begin{array}{c|c} R(-\pi/3) & \underbrace{R(-\pi/3)\begin{Bmatrix}(2+\sqrt{3})/2\\ 1/2\end{Bmatrix} - R(-\pi/3)\begin{Bmatrix}(1+\sqrt{3})/2\\ (1+\sqrt{3})/2\end{Bmatrix}} \\ \hline 0 & 1\end{array}\right]$$

$$= \left[\begin{array}{c|c} R(-\pi/3) & R(-\pi/3)\begin{Bmatrix}3/2\\ -\sqrt{3}/2\end{Bmatrix} \\ \hline 0 & 1\end{array}\right]$$

$\Rightarrow$

$$g = \left[\begin{array}{cc} R(-\pi/3) & \begin{Bmatrix}0\\ -2\sqrt{3}\end{Bmatrix} \\ \hline 0 & 1\end{array}\right] \cong \underline{(0, -2\sqrt{3}, -\pi/3)}$$

Suppose that there is an object in the end-effector's grip.




The transformation from the end-effector to the object frame is:

$$h = (1/4, 0, 0) \cong \left[\begin{array}{c|c} I & \begin{matrix}1/4\\ 0\end{matrix} \\ \hline 0 & 1\end{array}\right]$$

How does the object transform?

Well, we want to go from

$$\text{object} \xrightarrow{h^{-1}} \underset{\text{initial configuration}}{\text{end-effector}} \xrightarrow{g} \underset{\text{final configuration}}{\text{end-effector}} \xrightarrow{h} \text{object}$$

We need to use the adjoint. Recall its definition:

$$Ad_{g_1} g_2 = g_1 g_2 g_1^{-1}$$

so, what we want is:

$$Ad_{h^{-1}} g = h^{-1} g (h^{-1})^{-1} = h^{-1} g h$$

$\Rightarrow$

$$g_{object} = Ad_{h^{-1}} g$$

what is $h^{-1}$? It is $h^{-1} = (-1/4, 0, 0)$

$\Rightarrow$

$$g_{object} = \overset{h^{-1}}{(-1/4, 0, 0)} \cdot \overset{g}{(0, -2\sqrt{3}, -\pi/3)} \cdot \overset{h}{(1/4, 0, 0)}$$

$$= \left[\begin{array}{c|c} I & \begin{matrix} -1/4 \\ 0 \end{matrix} \\ \hline 0 & 1 \end{array}\right] \cdot \left[\begin{array}{c|c} R(-\pi/3) & \begin{matrix} 0 \\ -2\sqrt{3} \end{matrix} \\ \hline 0 & 1 \end{array}\right] \cdot \left[\begin{array}{c|c} I & \begin{matrix} 1/4 \\ 0 \end{matrix} \\ \hline 0 & 1 \end{array}\right]$$

$$g_{object} = \left[\begin{array}{c|c} R(-\pi/3) & \begin{matrix} -1/8 \\ (2+\frac{1}{8})\sqrt{3} \end{matrix} \\ \hline 0 & 1 \end{array}\right] \cong \left(-\frac{1}{8}, (2+\frac{1}{8})\sqrt{3}, -\frac{\pi}{3}\right)$$

- One goal of class then figure out how to design trajectory for manipulator to follow to effect the transformation $g$, and how control enters into the picture.

OUTLINE :

1) composition of $SE(2) = E(2) \ltimes SO(2)$
2) $so(3)$
3) $SE(3) = E(3) \ltimes SO(3)$

## Composition of SE(2)

recall that homogeneous representation of $SE(2)$ is:

ask class $\mapsto$ $g = \left[\begin{array}{c|c} R & \vec{d} \\ \hline 0 & 1 \end{array}\right]$

which accounted for (1) translation in space
(2) orientation of coordinate frame.

translation in space is given by $E(2)$ - Euclidean 2space.

what about the rotations? note that

$$R(\theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

and

$$R^{-1}(\theta) = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix} = R^T(\theta)$$

$$\|$$

$$\begin{bmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(\theta) \end{bmatrix} = R(-\theta)$$

since $R^{-1}(\theta) = R^T(\theta)$
$\Rightarrow$

$$R(\theta) R^T(\theta) = R^T(\theta) R(\theta) = I$$

so, $R(\theta)$ is an orthogonal matrix.

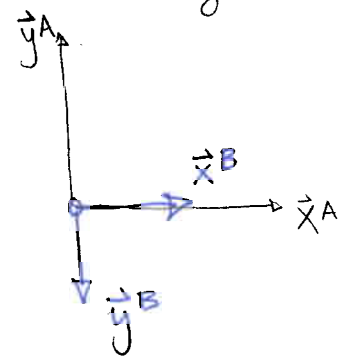
Q: can R be any orthogonal matrix? No!
↑ ask class

Here is a bad orthogonal matrix:

$$\begin{bmatrix} \cos(\theta) & \sin(\theta) \\ \sin(\theta) & -\cos(\theta) \end{bmatrix}$$

Why is it bad? Plot the axes

for $\theta = 0$, we get $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} | & | \\ \vec{x}^B & \vec{y}^B \\ | & | \end{bmatrix}$




compare to

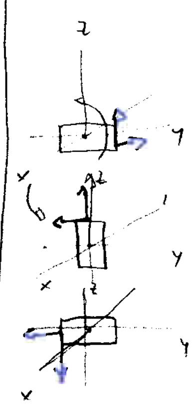
$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} | & | \\ \vec{x}^A & \vec{y}^A \\ | & | \end{bmatrix}$$

we have reversed orientation!
it is now a left-hand rule.

- This is equivalent to flipping an object about the x-axis

$\Rightarrow$

such a motion would violate the definition of planar motion! can you see why? $\rightarrow$




point follows circular motion outside of plane!

(use whiteboard eraser to visualize)

note that $\det\left(\begin{bmatrix} \cos(\theta) & \sin(\theta) \\ \sin(\theta) & -\cos(\theta) \end{bmatrix}\right) = -1$

$$\det\left(\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}\right) = 1$$

So, we see that the rotation matrix is a specific type of element in $O(2)$

**Definition.** $O(2)$ - the Lie group consisting of orthogonal $2 \times 2$ matrices, called the orthogonal group in 2-space

It is an element of $SO(2)$

**Definition** $SO(2)$ - special orthogonal group

the subspace of $O(2)$ with unit determinat, e.g., $A \in SO(2) \Rightarrow \det(A) = +1$.

Def.

~~The same holds for SE(3). Let's see how.~~

to recap $SE(2)$ has two parts, one in $E(2)$
one in $SO(2)$

$\Rightarrow$

we say that $SE(2) = E(2) \times SO(2)$

called the direct product.

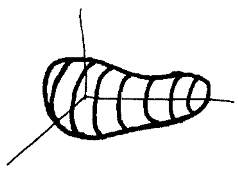
Q: What do you think $SE(3)$ looks like?

A: $SE(3) = E(3) \times SO(3)$

Let's see how.

## THE SPACE SO(3).

Consider a rigid body in E(3).



Let's play the similar trick as before, but keep object frame fixed at origin.

The rotation matrix R can realign the object, but cannot change distance between particles in body

$\Rightarrow$

$$\| p_i - p_j \| = \| R p_i - R p_j \| , \quad p_i, p_j \text{ are particles.}$$

where $\| \cdot \|$ denotes Euclidean distance/norm,

$$\| \vec{x} \| = \sqrt{\vec{x}^T \vec{x}}$$

$\Rightarrow$

$$\| \vec{v} \| = \| R(p_i - p_j) \| = \| R \vec{v} \| \quad \text{where } \vec{v} = p_i - p_j .$$

$\Rightarrow$

$$\| \vec{v} \|^2 = \| R \vec{v} \|^2$$

$\Rightarrow$

$$\vec{v}^T \vec{v} = (R\vec{v})^T R \vec{v} = \vec{v}^T R^T R \vec{v}$$

$\Rightarrow$

$$R^T R = I$$

$\Rightarrow$

R is orthogonal

but, recall that $\det(R) = \pm 1$ if R orthogonal.

since we cannot change right-hand to left-hand rule

$\Rightarrow$

$$\det(R) = 1$$

Thus $R \in SO(3)$

special orthogonal group in 3-space.

Craig calls this space of proper orthogonal matrices

## THE SPACE SE(3)

If we incorporate translation into the mix:

$$SE(3) = E(3) \times SO(3)$$

Note how the representation in homogeneous coordinates is still

$$\left[\begin{array}{c|c} R & \vec{d} \\ \hline 0 & 1 \end{array}\right]$$

except now

$$R = [3 \times 3] \,, \quad \vec{d} = [3 \times 1]$$

$\Rightarrow$

$$\left[\begin{array}{c|c} R & \vec{d} \\ \hline 0 & 1 \end{array}\right] = \left[\begin{array}{c|c} 3\times 3 & 3\times 1 \\ \hline 1\times 3 & 1\times 1 \end{array}\right] = [4 \times 4]$$

In general, $SE(n) = E(n) \times SO(n)$

and

$$g = \left[\begin{array}{c|c} R & \vec{d} \\ \hline 0 & 1 \end{array}\right] = \left[\begin{array}{c|c} n\times n & n\times 1 \\ \hline 1\times n & 1\times 1 \end{array}\right] = [(n+1) \times (n+1)]$$