

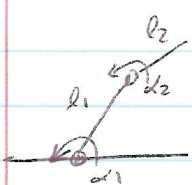
The Geometric Approach:

by observation, reduce geometry of the linkage to easily solved sub-problems

— one technique reduces the linkage to triangle.

Example:

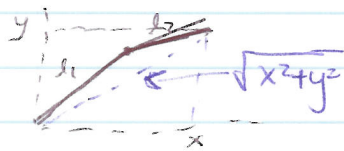
Many manipulators have the following sub-problem, which is usually obtained by projecting the manipulator geometry to 2 coordinate axes.



$$\begin{Bmatrix} x \\ y \\ \theta \end{Bmatrix} = \begin{Bmatrix} l_1 \cos \alpha_1 + l_2 \cos (\alpha_1 + \alpha_2) \\ l_1 \sin \alpha_1 + l_2 \sin (\alpha_1 + \alpha_2) \\ \alpha_1 + \alpha_2 \end{Bmatrix}$$

for this sub-problem only have 2 degrees of freedom  
 $\Rightarrow$  cannot place with arbitrary position & orientation.

$\Rightarrow$  so, just worry about positional placement.



To have solution, check:

$$l_1 + l_2 \geq \sqrt{x_{\text{des}}^2 + y_{\text{des}}^2} \geq |l_1 - l_2|$$

$\uparrow$  desired

(desired position should lie within annulus defined by reachable workspace for this subproblem)

If all is OK, then use law of cosines

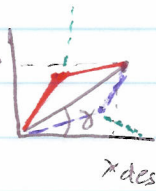


$$\alpha_1 = \beta + \gamma$$

$$\alpha_2 = \pi - \delta$$

$\Rightarrow$

Law of cosines:  $r^2 = l_1^2 + l_2^2 - 2l_1 l_2 \cos(\delta)$   
 $\Rightarrow \cos(\delta) = \frac{l_1^2 + l_2^2 - r^2}{2l_1 l_2}$

for  $\delta$ , we have:   $\Rightarrow \delta = \text{atan}(x_{des}, y_{des})$

note: atan is  
NOT atan2.  
for atan2, use  
 $\delta = \text{atan2}(y_{des}, x_{des})$

for  $\beta$ , use law of cosines with different ordering:

$$l_2^2 = l_1^2 + r^2 - 2l_1 r \cos(\beta)$$

$$\Rightarrow \cos(\beta) = \frac{l_1^2 + r^2 - l_2^2}{2l_1 r}$$

There is a little problem. Inverse of cosine is not unique (as shown in above figure, there are 2 shown ways to get to  $x_{des}$  and  $y_{des}$ ) (red and blue)

There are four possible solutions, two of which actually work.

So, solution obtained from:

$$\alpha_1 = \delta + \beta, \quad \delta - \beta$$

$$\alpha_2 = \pi - \delta, \quad \pi + \delta$$

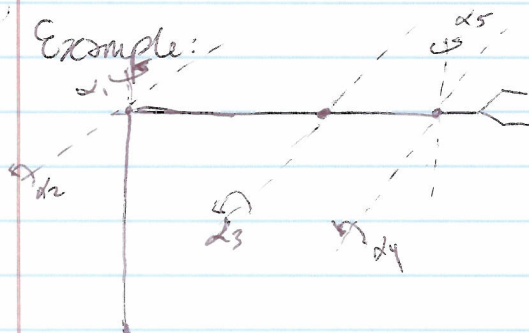
To Check: verify orientation

$$\alpha_1 + \alpha_2 = \delta + \beta + \pi - \delta$$

$$\delta - \beta + \pi + \delta$$

Just pick one of the two valid ones.

Example:

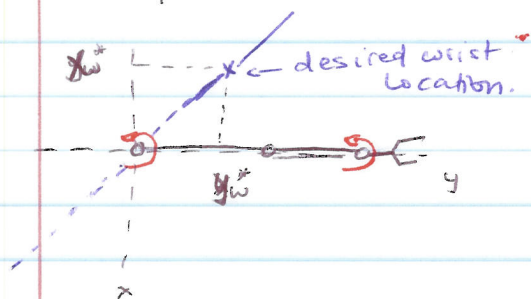


How to apply to more complex example.

- 1) break the problem down to
  - i) position of wrist
  - ii) orientation of hand/wrist
  - iii) angle of first joint.

assume (ii) solved enough to give us desired positioning wrist.

① Top view



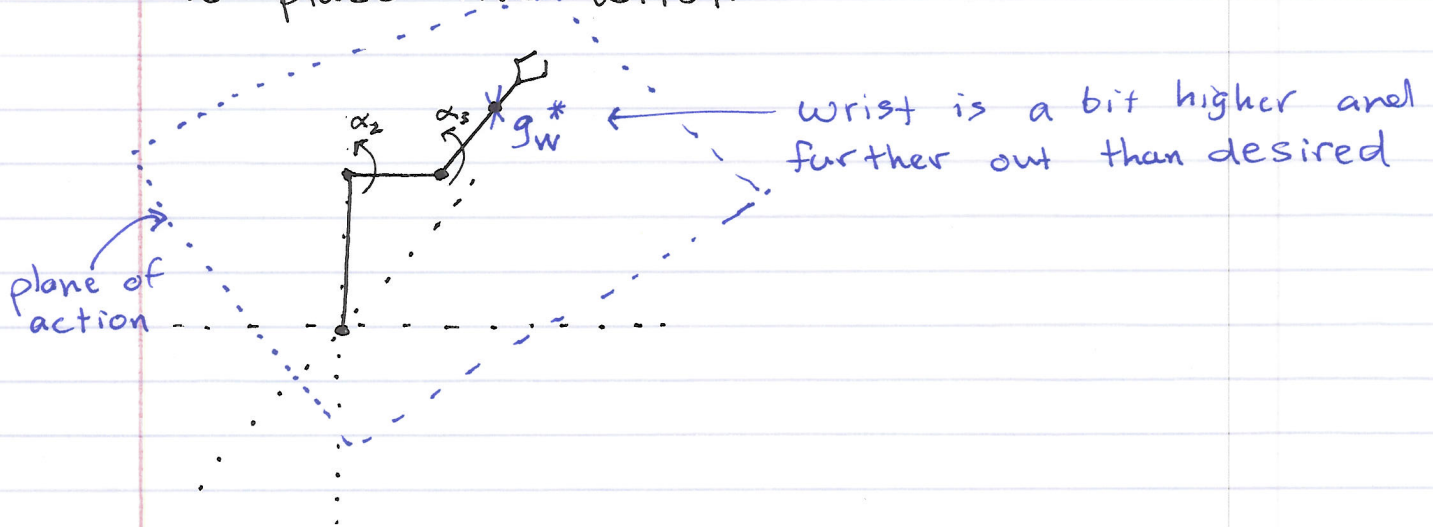
Solution is

$$\alpha_1 = \tan^{-1}(x_w^*, y_w^*) - \frac{\pi}{2}$$

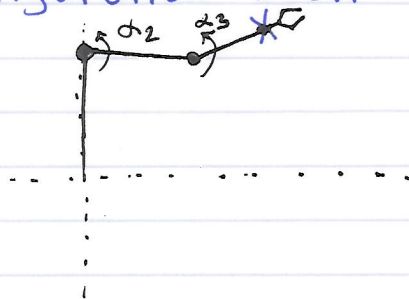
since reference configuration is aligned with w/ y-axis & not x-axis.

to be continued...

last time we placed the manipulators "plane of action" to include the point where we would like to place the wrist.

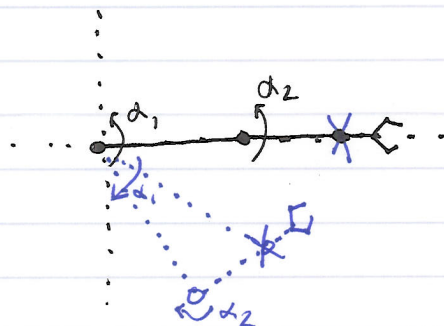


configuration seen in plane

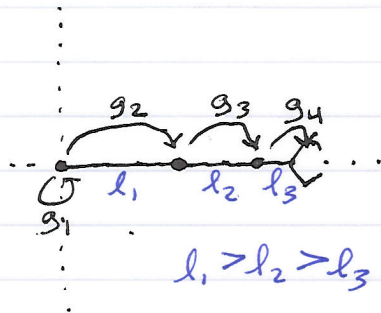


placement of only the position of the wrist reduces to the previous subproblem, with an appropriate coordinate change

shift origin up



Example Consider the planar 3R manipulator



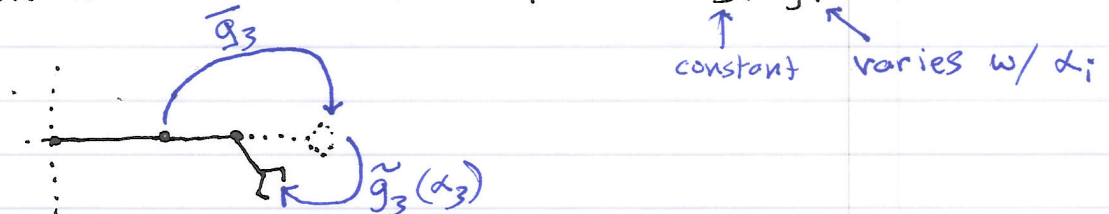
Here we have full of orientation inside dextous workspace

We have a defined end-effector config.  $g_e^*$

$$g_e(\alpha) = g_1(\alpha_1) g_2(\alpha_2) g_3(\alpha_3) g_4$$

$\uparrow$   
constant

Each  $g_i(\alpha_i)$  can be broken up into  $\bar{g}_i \tilde{g}_i(\alpha_i)$



$$g_e(\alpha) = \underbrace{g_1(\alpha_1) g_2(\alpha_2) \bar{g}_3}_{\text{moves to wrist}} \underbrace{\tilde{g}_3(\alpha_3) g_4}_{\text{wrist to gripper}} \quad (*)$$

Goal is to find an  $\alpha^*$  such that  $g_e^* = g_e(\alpha^*)$

$\Rightarrow$  Use equation (\*) above to break this down into 2 problems

find  $\alpha$  such that

$$g_e^* = g_e(\alpha) = g_1(\alpha_1) g_2(\alpha_2) \bar{g}_3 \tilde{g}_3(\alpha_3) g_4$$

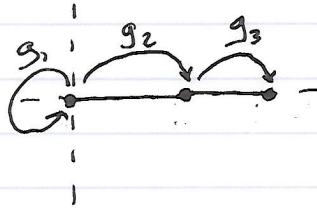
$$g_e^* g_4^{-1} = g_1(\alpha_1) g_2(\alpha_2) \bar{g}_3 \underline{\underline{\tilde{g}_3(\alpha_3)}}$$

$\hookrightarrow$  does not change my position!

$$\text{position}(g_e^* g_4^{-1}) = \text{position}(g_1(\alpha_1) g_2(\alpha_2) \bar{g}_3)$$

⇒ First Problem

Use triangle solution to find  $\alpha_1$  and  $\alpha_2$



After this, solve for orientation.