Direct MRAC for SISO Systems

- Let's make our first baby step away from strictly scalar systems to vectorsystems. Still, though, we have one input and one main output although we will utilize full-state feedback.

Plant/System Model:

parametrized nonlinearity

$$\dot{x}(t) = A\,x(t) + b\lambda\left(u(t) + \alpha^T \Phi(x(t))\right), \quad x(0) = x_0$$

unknowns

known

$A \in \mathbb{R}^{n\times n}$ (matrix)

$\text{sign}(\lambda)$ is known.

Reference Model:

$$\dot{x}_m(t) = A_m\, x_m(t) + b_m(t)\, r(t) \qquad x_m(0) = x_{m,0}$$

Hurwitz

cts. unif. bdd input

Controller:

$$u(t) = k_x^T(t)\, x(t) + k_r(t)\, r(t) - \hat{\alpha}^T(t)\, \Phi(x(t))$$

vector of adaptive gains.

substitution

$$\dot{x}(t) = \left[A + b\lambda k_x^T(t)\right] x(t) + b\lambda k_r(t) r(t) - b\lambda\, \Delta\alpha^T(t)\, \Phi(x(t))$$

MATCHING CONDITIONS:

$$A_m - A = b\lambda (k_x^*)^T$$

$$b_m = b\lambda k_r^*$$

Assuming the existence of a solution to the matching conditions, continue by analyzing error and error dynamics.

$$e(t) = x(t) - x_m(t)$$

$\Rightarrow$

$$\dot{e}(t) = \dot{x}(t) - \dot{x}_m(t)$$

$$= A_m e(t) + b\lambda \left[\Delta k_x^T(t)\, x(t) + \Delta k_r(t)\, r(t) - \Delta\alpha^T(t)\, \Phi(x(t)) \right]$$

where

$$\Delta k_x(t) = k_x(t) - k_x^*$$

$$\Delta k_r(t) = k_r(t) - k_r^*$$

$$\Delta\alpha(t) = \hat{\alpha}(t) - \alpha$$

choose candidate Lyapunov function

$$V(e(t), \Delta k_x(t), \Delta k_r(t), \Delta\alpha) = e^T(t) P e(t) + |\lambda| \left[\Delta k_x^T(t)\, \Gamma_x^{-1} \Delta k_x(t) + \gamma_r^{-1} \Delta k_r^2(t) + \Delta\alpha^T(t)\, \Gamma_\alpha^{-1} \Delta\alpha(t) \right]$$

where P satisfies $\underbrace{A_m^T P + P A_m}_{\text{symmetric.}} = -Q < 0$

P is pos. def. & symmetric

Such a P is guaranteed to exist for Q pos. def. symmetric since A_m is Hurwitz.

Take time derivative,

$$\dot{V}(t) = -e^T(t)Qe(t) + \overbrace{2e^T(t)Pb}^{\text{scalar}}\lambda\left[\Delta k_x^T(t)x(t) + \Delta k_r(t)r(t) - \Delta\alpha^T(t)\Phi(x(t))\right]$$
$$+ 2|\lambda|\,\Delta k_x^T(t)\Gamma_x^{-1}\dot{\Delta k}_x(t) + 2|\lambda|\gamma_r^{-1}\Delta k_r(t)\dot{\Delta k}_r(t)$$
$$+ 2|\lambda|\,\Delta\alpha^T(t)\Gamma_\alpha^{-1}\dot{\Delta\alpha}(t)$$

$$= -e^T(t)Qe(t) + 2|\lambda|\,\Delta k_x^T(t)\left[x(t)e^T(t)Pb\,\mathrm{sign}(\lambda) + \Gamma_x^{-1}\dot{\Delta k}_x(t)\right]$$
$$+ 2|\lambda|\,\Delta k_r(t)\left[r(t)e^T(t)Pb\,\mathrm{sign}(\lambda) + \gamma_r^{-1}\dot{\Delta k}_r(t)\right]$$
$$+ 2|\lambda|\,\Delta\alpha^T(t)\underbrace{\left[-\Phi(x(t))e^T(t)Pb\,\mathrm{sign}(\lambda) + \Gamma_\alpha^{-1}\dot{\Delta\alpha}(t)\right]}_{\text{want bracketted terms to vanish.}}$$

define the updates,

$$\left.\begin{aligned}
\dot{\Delta k}_x(t) &= \dot{k}_x(t) \equiv -\Gamma_x x(t)e^T(t)Pb\,\mathrm{sign}(\lambda)\\
\dot{\Delta k}_r(t) &= \dot{k}_r(t) \equiv -\gamma_r r(t)e^T(t)Pb\,\mathrm{sign}(\lambda)\\
\dot{\Delta\alpha}(t) &= \dot{\hat{\alpha}}(t) \equiv \Gamma_\alpha\Phi(x(t))e^T(t)Pb\,\mathrm{sign}(\lambda)
\end{aligned}\right\}\ \underline{\text{Adaptation}}$$

$\Rightarrow$

$$\dot{V}(t) = -e^T(t)Qe(t) \le 0 \quad \text{negative semi-definite}$$

from which we get stability and boundedness of signals.

Barbalat's Lemma allows us to conclude that $e(t) \to 0$ as $t \to \infty$, under the appropriate conditions.

MIMO Precursors:

- Before moving on to MIMO systems, need to cover a few precursors.

Working with MIMO systems will necessarily involve working with matrices and also norms on matrices. We could try to use the induced norms that we are used to, but they have their issues.

Induced norm on a matrix operates as follows

$$\|A\| = \underbrace{\sup_{\|x\|=1} \|Ax\|}$$

now, 1) how this works varies w/choice of norm
2) differentiability may not be easy or existent.

EXAMPLES.

1) $\|x\|_\infty = \max_i |x_i| \Rightarrow \|A\|_\infty = \max_i \left| \sum_j a_{ij} \right|$

2) $\|x\|_1 = \sum |x_i| \Rightarrow \|A\|_1 = \max_j \sum_i |a_{ij}|$

3) $\|x\|_2 = \left(\sum_i x_i^2 \right)^{1/2} \Rightarrow \|A\|_2 = \underbrace{\max_i \sigma_i}_{\text{maximum singular value of } A.}$

Let's use another norm for matrices that will be easier to deal with. It is not an induced norm. It is called the Frobenius norm,

$$\|A\|_F = \operatorname{tr}(A^TA) = \sum_{ij} a_{ij}^2$$

trace operator

note:

$$A^T A \;\Rightarrow\; \operatorname{tr}(A^TA) = \sum_i c_i \cdot c_i = \sum_i c_i^T c_i$$

columns of A.

- trace is defined to be the sum of the diagonal elements

The Frobenius norm can be modified using Γ, positive definite & symmetric,

$$\|A\|_{F,\Gamma} = \operatorname{tr}(A^T\Gamma A)$$

It also satisfies the following identities:

- $a^Tb = \operatorname{tr}(ba^T)$ for vectors $a, b \in \mathbb{R}^n$.
- Frobenius inner product: $\langle A, B\rangle_F = A^TB$

$$\|\langle A,B\rangle_F\|_F = \|A^TB\|_F \le \|A\|_F \, \|B\|_F$$

- note, can also do $\langle A,B\rangle_{F,\Gamma} = A^T\Gamma B$ $\quad \Gamma = \Gamma^T > 0$.

Direct MRAC, MIMO Systems:

Plant/System Model: $\dot{x}(t) = A\,x(t) + B\Lambda\left(u(t) + f(x(t))\right)$

unknowns

known matrix

Λ diagonal, non-negative.

$f(x)$ parametrizable.

$$A \in \mathbb{R}^{n\times n}, \quad B \in \mathbb{R}^{n\times m}, \quad \Lambda \in \mathbb{R}^{m\times m}$$

* Λ can be used to model control failure.
* $(A, B\Lambda)$ should be controllable.
* $f(x)$ represents matched uncertainty, $f(x) = \alpha^T \Phi(x)$

  (usually leftover parts unaccounted for by linearization but that are important)

  - often called matched structured uncertainty.

Reference Model: $\dot{x}_m(t) = A_m\,x_m(t) + B_m r(t)$

Hurwitz

cts. unif. bdd.

Controller: $u(t) = K_x^T(t)\,x(t) + K_r^T(t)\,r(t) - \hat{\alpha}^T(t)\,\Phi(x(t))$

$$K_x \in \mathbb{R}^{n\times m}, \quad K_r \in \mathbb{R}^{m\times m}, \quad \hat{\alpha} \in \mathbb{R}^{N\times m}$$

⇒ substitute & compare to model

$$A - A_m = B\Lambda (K_x^*)^T$$
$$B_m = B\Lambda (K_r^*)^T$$
} MATCHING CONDITIONS.

- uncertainty in A w/respect to model needs to be in the range-space of B (or $B\Lambda$, if some entries of Λ can vanish on the diagonal).

- matching conditions are a little bit more complicated & a solution may not exist.

  no solution ⇒ controller does not have structure required to meet control objectives.

- usually there is enough information about A & B to design the proper (A_m, B_m) pair. (Have an example soon.)

(closed-loop system

$$\dot{x}(t) = (A + B\Lambda K_x^T(t))\, x(t) + B\Lambda\left[K_r^T(t)\, r(t) - \Delta\alpha^T(t)\, \Phi(x(t))\right]$$

↳ error + error dynamics

$$e(t) = x(t) - x_m(t)$$

$$\dot{e}(t) = \dot{x}(t) - \dot{x}_m(t)$$

$$= A_m e(t) + B\Lambda\left[\Delta K_x^T(t)\, x(t) + \Delta K_r^T(t)\, r(t) - \Delta\alpha^T(t)\, \Phi(x(t))\right]$$

next up: The Lyapunov function (candidate, of course).

before we had used $\Delta k_x^T(t)\Gamma_x^{-1}\Delta k_x(t)$ and/or $\gamma_r \Delta k_r^2(t)$

for adaptive term components of the (candidate) Lyapunov function.

Now that these are matrices, we need to consider matrix inner products/norms.

We are going to use a modified version of the Frobenius norm.

$$V(e(t), \Delta K_x(t), \Delta K_r(t), \Delta\alpha(t))$$

$$= e^T(t) P e(t) + tr\left(\left[\Delta K_x^T(t) \Gamma_x^{-1} \Delta K_x(t) + \Delta K_r^T(t) \Gamma_r^{-1} \Delta K_r(t) + \Delta\alpha^T(t) \Gamma_\alpha^{-1} \Delta\alpha(t) \right] \Delta \right)$$

- we have existence of Q so that $A_m^T P + P A_m = -Q < 0$
- matrix norm parts arise from $\|\Delta K\|_{F,\Gamma^{-1}\Delta} = tr(\Delta K^T \Gamma^{-1} \Delta \Delta K)$

$$= tr(\Delta K^T \Gamma^{-1} \Delta K \Delta)$$

since Δ is diagonal,

and also linearity of tr, $tr(M_1 + M_2) = tr(M_1) + tr(M_2)$.

$$\Rightarrow \quad \dot V = \left(A_m e(t) + B\Delta \left[\Delta K_x^T(t) x(t) + \Delta K_r^T(t) r(t) - \Delta\alpha^T(t) \Phi(x(t)) \right] \right)^T P e(t)$$

$$+ e^T(t) P \left(A_m e(t) + B\Delta \left[\Delta K_x^T(t) x(t) + \Delta K_r^T(t) r(t) - \Delta\alpha^T(t) \Phi(x(t)) \right] \right)$$

$$+ 2 tr\left(\left[\Delta K_x^T(t) \Gamma_x^{-1} \dot{\Delta K}_x(t) + \Delta K_r^T \Gamma_r^{-1} \dot{\Delta K}_r + \Delta\alpha^T \Gamma_\alpha^{-1} \dot{\Delta\alpha} \right] \Delta \right)$$

$$= e^T(t) \left(A_m^T P + P A_m \right) e(t)$$

$$+ \left(B\Delta \left[\Delta K_x^T(t) x(t) + \Delta K_r^T(t) r(t) - \Delta\alpha^T(t) \Phi(x(t)) \right] \right)^T P e(t)$$

$$+ e^T(t) P \left(B\Delta \left[\Delta K_x^T(t) x(t) + \Delta K_r^T(t) r(t) - \Delta\alpha^T(t) \Phi(x(t)) \right] \right)$$

$$+ 2 tr\left(\left[\Delta K_x^T(t) \Gamma_x^{-1} \dot{\Delta K}_x(t) + \Delta K_r^T(t) \Gamma_r^{-1} \dot{\Delta K}_r + \Delta\alpha^T(t) \Gamma_\alpha^{-1} \dot{\Delta\alpha}(t) \right] \Delta \right)$$

$$\dot V = -e^T(t)\,Q\,e(t) + 2e^T(t)PB\Lambda\left[\Delta K_x^T(t)\,x(t) + \Delta K_r^T(t)\,r(t) - \Delta\alpha^T(t)\,\Phi(x(t))\right]$$
$$+\,2\,tr\left(\left[\Delta K_x^T(t)\Gamma_x^{-1}\dot{\Delta K}_x(t) + \Delta K_r^T(t)\Gamma_r^{-1}\dot{\Delta K}_r(t) + \Delta\alpha^T(t)\Gamma_\alpha^{-1}\dot{\Delta\alpha}(t)\right]\Lambda\right)$$

$$= -e^T(t)\,Q\,e(t) + 2\left[e^T(t)PB\Lambda\,\Delta K_x^T(t)\,x(t) + tr\left(\Delta K_x^T(t)\Gamma_x^{-1}\dot{\Delta K}_x(t)\Lambda\right)\right]$$
$$+\,2\left[e^T(t)PB\Lambda\,\Delta K_r^T(t)\,r(t) + tr\left(\Delta K_r^T(t)\Gamma_r^{-1}\dot{\Delta K}_r(t)\Lambda\right)\right]$$
$$+\,2\left[\underbrace{\underbrace{e^T(t)PB\Lambda}_{a^T}\,\underbrace{\Delta\alpha^T(t)\,\Phi(x(t))}_{b}}_{=\,tr(ba^T)} + tr\left(\Delta\alpha^T(t)\Gamma_\alpha^{-1}\dot{\Delta\alpha}(t)\Lambda\right)\right]$$

$$= -e^T(t)\,Q\,e(t) + 2\,tr\left(\Delta K_x^T(t)\left[x(t)\,e^T(t)PB + \Gamma_x^{-1}\dot{\Delta K}_x(t)\right]\Lambda\right)$$
$$+\,2\,tr\left(\Delta K_r^T(t)\left[r(t)\,e^T(t)PB + \Gamma_r^{-1}\dot{\Delta K}_r(t)\right]\Lambda\right)$$
$$+\,2\,tr\left(\Delta\alpha^T(t)\underbrace{\left[\Phi(x(t))e^T(t)PB + \Gamma_\alpha^{-1}\dot{\Delta\alpha}(t)\right]}_{\text{want bracketted terms to vanish}}\Lambda\right)$$

define:

$$\left.\begin{aligned}
\dot{\Delta K}_x(t) &= \dot K_x(t) \equiv -\Gamma_x\,x(t)\,e^T(t)PB\\
\dot{\Delta K}_r(t) &= \dot K_r(t) \equiv -\Gamma_r\,r(t)\,e^T(t)PB\\
\dot{\Delta\alpha}(t) &= \dot{\hat\alpha}(t) \equiv \Gamma_\alpha\,\Phi(x(t))\,e^T(t)PB
\end{aligned}\right\}\ \underline{\text{Adaptation}}$$

$\Rightarrow$

$$\dot V = -e^T(t)\,Q\,e(t) \le 0 \quad \text{negative semi-definite}$$

if Λ must have non-negative and non-positive elements, then the Λ term in V becomes $|\Lambda|$ and the adaptation laws become

$$\dot{K}_x(t) \equiv -\Gamma_x\, x(t)\, e^T(t)\, PB\, \mathrm{sign}(\Lambda)$$

$$\dot{K}_r(t) \equiv -\Gamma_r\, r(t) e^T(t) PB\, \mathrm{sign}(\Lambda)$$

$$\dot{\hat{\alpha}}(t) \equiv \Gamma_\alpha\, \Phi(x(t))\, e^T(t) PB\, \mathrm{sign}(\Lambda)$$

where,

$$|\Lambda| = \mathrm{diag}(|\lambda_1|, \ldots, |\lambda_m|)$$

$$\mathrm{sign}\,\Lambda = \mathrm{diag}(\mathrm{sign}(\lambda_1), \ldots, \mathrm{sign}(\lambda_m))$$

* recall Λ defined to be $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_m)$.

Augmentation of a Nominal Design (LQR Case)

How does one get A_m and b_m?

- b_m is typically set by the control structure of the system.
  - A_m is obtained through some design procedure
    - for a linear system one can perform LQR design to create a stable system w/some desired performance.

↳ but then, how does one incorporate adaptation in a stable way ensuring the proper transition from the nominal to the adaptive system?

Let's do this for a SISO system and leave the MIMO case to the reader (YOU!).

1) Start with a nominal plant to do design on:

Nominal plant: $\dot{x}_n(t) = A_0 x_n(t) + b_0 u(t) \qquad x(0) = x_0$

LQR leads to

$$u_{LQR}(t) = -k_{LQR}^T x(t) + k_{REF}\, r(t)$$

apply LQR state-feedback

$$\dot{x}_m(t) = \underbrace{A_m}_{\equiv\, A_0 - b_0 k_{LQR}^T} x_m(t) + \underbrace{b_m}_{\equiv\, b_0 k_{REF}} r(t)$$

2) Figure out adaptive controller structure given plant:

Plant|System Model: $$x(t) = Ax(t) + b\lambda\left[u(t) + \alpha^T \Phi(x(t))\right]$$

unknowns

known

our control will be $$u(t) = u_{LQR}(t) + u_{ad}(t)$$

where $$u_{ad}(t) = k_x^T(t)\,x(t) + k_r(t)\,r(t) - \hat{\alpha}^T(t)\,\Phi(x(t))$$

↓ substitute to determine matching conditions

$$\dot{x}(t) = \left(A - b\lambda k_{LQR}^T + b\lambda k_x^T(t)\right)x(t) + b\lambda\left(k_{REF} + k_r(t)\right)r(t) - b\lambda\,\Delta\alpha^T(t)\,\Phi(x(t))$$

↓ compare against model

$$A + b\lambda\left(k_x(t) - k_{LQR}\right)^T = A_o - b_o k_{LQR}^T$$

$$b\lambda\left(k_r(t) + k_{REF}\right) = b_o k_{REF}$$

↓ equivalent to

$$A + b\lambda(k_x^*)^T = A_o - b_o k_{LQR}^T$$

$$b\lambda k_r^* = b_o k_{REF}$$

↓

$$b\lambda(k_x^*)^T = A_o - A - bk_{LQR}^T$$

$$b\lambda k_r^* = b_o k_{REF}$$

MATCHING CONDITIONS

Once the matching conditions have been verified, the standard procedure must now be followed

$$\dot{x}(t) = \left(A - b\lambda k_{LQR}^T + b\lambda k_x^T(t)\right) x(t) + b\lambda\left(k_{REF} + k_r(t)\right) r(t) - b\lambda\, \Delta\alpha^T(t)\,\Phi(x(t))$$

$$= \left(A + b\lambda (k_x^*)^T - b\lambda (k_x^*)^T - b\lambda k_{LQR}^T + b\lambda k_x^T(t)\right) x(t)$$

$$+ b\lambda\left(k_r^* - k_r^* + k_{REF} + k_r(t)\right) r(t) - b\lambda\, \Delta\alpha^T(t)\,\Phi(x(t))$$

$$= A_m x(t) + b_m r(t) + b\lambda\left(k_x(t) - k_{LQR} - k_x^*\right)^T x(t)$$

$$+ b\lambda\left(k_r(t) - k_{REF} - k_r^*\right) r(t)$$

$$- b\lambda\, \Delta\alpha^T(t)\,\Phi(x(t))$$

$$= A_m x(t) + b_m r(t) + b\lambda\left[\Delta k_x^T(t)\, x(t) + \Delta k_r(t)\, r(t) - \Delta\alpha^T(t)\,\Phi(x(t))\right]$$

$$\Delta k_x(t) = k_x(t) - k_{LQR} - k_x^*$$

$$\Delta k_r(t) = k_r(t) - k_{REF} - k_r^*$$

$$\Delta\alpha(t) = \hat{\alpha}(t) - \alpha$$

Leads to error and error dynamics

$$e(t) = x(t) - x_m(t)$$

$$\dot{e}(t) = \dot{x}(t) - \dot{x}_m(t)$$

$$= A_m e(t) + b\lambda\left[\Delta k_x^T(t)\, x(t) + \Delta k_r(t)\, r(t) - \Delta\alpha^T(t)\,\Phi(x(t))\right]$$

- this is pretty much the same error as before.

  Makes sense since we are not doing anything special.

... now, on to the Lyapunov part ...

choose

$$V(e(t), \Delta k_x(t), \Delta k_r(t), \Delta\alpha(t)) = e^T(t) P e(t) + |\lambda| \left[\Delta k_x^T(t) \Gamma_x^{-1} \Delta k_x(t) + \gamma_r^{-1} \Delta k_r^2(t) + \Delta\alpha^T(t) \Gamma_\alpha^{-1} \Delta\alpha(t) \right]$$

↳ leads to

$$\begin{aligned} \dot{V} = \; & - e^T(t) Q e(t) \\ & \underbrace{\begin{aligned} & + 2|\lambda| \Delta k_x^T(t) \left[x(t) e^T(t) P b \, \mathrm{sign}(\lambda) + \Gamma_x^{-1} \Delta\dot{k}_x(t) \right] \\ & + 2|\lambda| \Delta k_r(t) \left[r(t) e^T(t) P b \, \mathrm{sign}(\lambda) + \gamma_r^{-1} \Delta\dot{k}_r(t) \right] \\ & + 2|\lambda| \Delta\alpha^T(t) \left[-\Phi(x(t)) e^T(t) P b \, \mathrm{sign}(\lambda) + \Gamma_\alpha^{-1} \Delta\dot{\alpha}(t) \right] \end{aligned}}_{\text{Want bracketted terms to vanish.}} \end{aligned}$$

define,

$$\Delta\dot{k}_x(t) = \dot{k}_x(t) = -\Gamma_x x(t) e^T(t) P b \, \mathrm{sign}(\lambda)$$

$$\Delta\dot{k}_r(t) = \dot{k}_r(t) = -\gamma_r r(t) e^T(t) P b \, \mathrm{sign}(\lambda)$$

$$\Delta\dot{\alpha}(t) = \dot{\alpha}(t) = \Gamma_\alpha \Phi(x(t)) e^T(t) P b \, \mathrm{sign}(\lambda)$$

$\Rightarrow$

$$\dot{V} = - e^T(t) Q e(t) \le 0 \quad \text{neg. semi-definite}$$

and the rest follows as before.

Some things to note

- this is augmentation of an already stable, but potentially suboptimal, control mechanism. Reason for degraded performance could be minor discrepancies between model and real plant, incorrect control gains (λ), or ~~the~~ (matched) nonlinearities.

- with no adaptation and adaptive parameters set properly, the system should operate using the pre-existing control law.

- adaptive ~~parameters~~ gains slowly tuned away from zero until desired performance achieved.

- it may be that the linear terms are well characterized. In which case $A = A_0$, and some of the uncertainty is no longer present.

Augmentation of a Nominal PI Controller (SISO System)

* addition of integral control term is standard for many systems.
  - integral term removes steady-state error

Plant | System Model:

$$\dot{x}(t) = A\,x(t) + b\,\lambda\left[u(t) + \alpha^T \Phi(x(t))\right]$$

$$y(t) = c^T x(t)$$

known quantities

unknowns
sign of λ known

- observe system through single output y in order to determine if it is tracking desired output $y_c(t)$.
- this is just to compare against a scalar signal. The full state can still be measured for state feedback.
- need $(A, b\lambda)$ controllable.

GOAL: Design full-state adaptive controller so that $y(t)$ tracks $y_c(t)$.

To get the integral term, define the error integral signal & augment the system.

$$\tilde{y}(t) = y(t) - y_c(t) = c^T x(t) - y_c(t)$$

error integral

$$y_I(t) = \int_0^t \tilde{y}(\tau)\,d\tau$$

The augmented system is :

$$\xi(t) = \begin{Bmatrix} y_I(t) \\ x(t) \end{Bmatrix}$$

$\Rightarrow$

$$\dot{\xi}(t) = \underbrace{\begin{bmatrix} 0 & c^T \\ 0 & A \end{bmatrix}}_{\bar{A}} \xi + \underbrace{\begin{bmatrix} 0 \\ b \end{bmatrix}}_{\bar{b}} \lambda \left[u(t) + \alpha^T \Phi(x(t)) \right] + \underbrace{\begin{bmatrix} -1 \\ 0 \end{bmatrix}}_{b_2} y_c(t)$$

$\Rightarrow$

$$\dot{\xi} = \bar{A} \xi + \bar{b} \lambda \left[u(t) + \alpha^T \Phi(x(t)) \right] + b_2 y_c(t)$$

this is the plant model.

Reference Model:

- design a feedback control law to achieve stabilization and asymptotic tracking of $y_c(t)$,

$$u_c(t) = -k_c^T \xi(t)$$

this becomes the model

$$\dot{\xi}_m(t) = \underbrace{(\bar{A} - \bar{b} k_c^T)}_{} \xi_m(t) + b_2 y_c(t)$$

$$\dot{\xi}_m(t) = A_m \xi_m(t) + b_2 y_c(t)$$

Controller:

The adaptive controller is:

$$u(t) = u_c(t) + u_{ad}(t)$$

where

$$u_{ad}(t) = k_\xi(t)\,\xi(t) - \hat{\alpha}^T(t)\,\Phi(x(t))$$

notice no $k_y(t) y_c(t)$.

why? it makes sense, . . .

Let's examine the error and error dynamics with $u(t) = u_c(t) + u_{ad}(t)$

$$e(t) \equiv \xi(t) - \xi_m(t)$$

$$\Rightarrow \dot{e}(t) = \dot{\xi}(t) - \dot{\xi}_m(t)$$

$$= \bar{A}_m e(t) + \bar{b} k_c^T \xi(t) + \bar{b}\lambda\left[u_{ad}(t) + \Delta\alpha^T(t)\,\Phi(x(t))\right]$$

error dynamics do not have the $b_2 y_c(t)$ term at all.

it cancelled out. this explains why no adaptive elements are needed for it (as an "artificial" addition to the system, we know its structure exactly).

no adaptation is needed for feedforward terms like this one.

$\Rightarrow$ continuing with the error dynamics

$$\dot{e}(t) = \bar{A}_m e(t) + \bar{b}\lambda \left[k_\xi^T(t)\xi(t) - \frac{1}{\lambda} k_c^T(t)\xi(t) - \Delta\alpha^T(t)\Phi(x(t)) \right]$$

- here it was assumed that A is known, the only uncertainty is in λ & α. This means that we seek to adapt to unknown control authority and nonlinear terms. Matching conditions not really needed since system structure is such that an ideal solution exists.

$\Rightarrow$

$$\dot{e}(t) = \bar{A}_m e(t) + \bar{b}\lambda \left[\Delta k_\xi^T(t)\xi(t) - \Delta\alpha^T(t)\Phi(x(t)) \right]$$

$$\Delta k_\xi(t) = k_\xi(t) - \frac{1}{\lambda} k_c$$

$$\Delta\alpha(t) = \hat{\alpha}(t) - \alpha$$

Using the candidate Lyapunov function

$$V(e(t), \Delta k_\xi(t), \Delta\alpha(t)) = e^T(t) P e(t) + |\lambda| \left[\Delta k_\xi^T(t) \Gamma_\xi^{-1} \Delta k_\xi(t) + \Delta\alpha^T(t) \Gamma_\alpha^{-1} \Delta\alpha(t) \right],$$

we get

$$\dot{V} = -e^T(t) Q e(t) + 2e^T(t) P \bar{b} \lambda \left[\Delta k_\xi^T(t)\xi(t) + \Delta\alpha^T(t)\Phi(x(t)) \right]$$
$$+ 2|\lambda| \left[\Delta k_\xi^T(t) \Gamma_\xi^{-1} \Delta\dot{k}_\xi(t) + \Delta\alpha^T(t) \Gamma_\alpha^{-1} \Delta\dot{\alpha}(t) \right]$$

$\Rightarrow$

$$\dot{V} = -e^T(t) Q e(t) + 2|\lambda| \Delta k_\xi^T(t) \left[\xi(t) e^T(t) P \bar{b} \, \mathrm{sign}(\lambda) + \Gamma_\xi^{-1} \Delta \dot{k}_\xi(t) \right]$$

$$+ 2|\lambda| \Delta \alpha^T(t) \left[\Phi(x(t)) e^T(t) P \bar{b} \, \mathrm{sign}(\lambda) + \Gamma_\alpha^{-1} \Delta \dot{\alpha}(t) \right]$$

want bracketted terms to vanish

define,

$$\Delta k_\xi(t) = \dot{k}_\xi(t) = -\Gamma_\xi \xi(t) e^T(t) P \bar{b} \, \mathrm{sign}(\lambda)$$

$$\Delta \alpha(t) = \dot{\hat{\alpha}}(t) = -\Gamma_\alpha \Phi(x(t)) e^T(t) P \bar{b} \, \mathrm{sign}(\lambda)$$

Adaptation

$\Rightarrow$

$$\dot{V} = -e^T(t) Q e(t) \leq 0$$ negative semi-definite

... and the rest follows ...