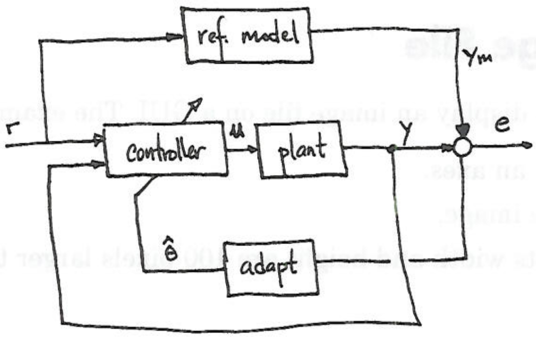
Model Reference Adaptive Control



System has four main components:

1] Plant : known structure with unknown parameters

2] Reference Model : specifies ideal/desired response of the system to external commands. it is part of the adaptive control system design.

It should

a) reflect performance specifications of the control task. (rise time, settle time, overshoot, etc.)

b) should be achievable for the adaptive control system w/ its structural characteristics.
(order, relative degree of regulated outputs, etc.)

3] Controller : feedback/feedforward control law with adjustable parameters

should have perfect tracking ability if parameters exactly known.

4] Adaptation Law : adjust parameters in the control law.

GOAL : make tracking error, e, converge to zero.

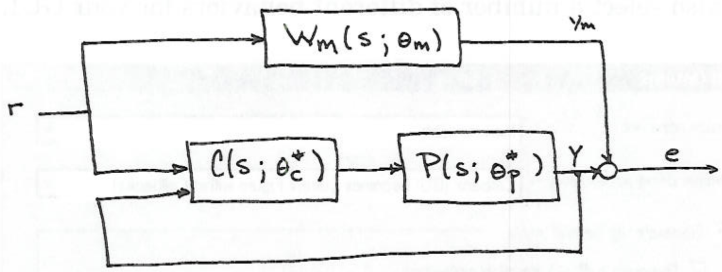
must gaurantee stable controller meanwhile.

Direct & Indirect MRAC

1) direct - adjust controller parameters during adaptation.

2) indirect - estimate plant parameters for use by controller.

What is the main principle? ... Well, in the frequency domain lets examine the following model reference controller,



$\theta_m, \theta_c, \theta_p$ - coefficients/parameters of the transfer function

$W_m(s;\theta_m)$ ← reference model achieving desired operating characteristics

Goal is to determine what θ_c^* should be. This is equivalent to finding a law ~~such that~~

$$\theta_c^* = F(\theta_p^*, \theta_m) \quad \text{such that} \quad \frac{Y(s)}{R(s)} = \frac{Y_m(s)}{R(s)}$$

if $W_m(s;\theta_m) = \dfrac{N_m(s;\theta_m)}{D_m(s;\theta_m)}$ then find θ_c^* so that $\dfrac{Y(s)}{R(s)} = \dfrac{PC}{1+PC} = \dfrac{N(s;\theta_p^*,\theta_c^*)}{D(s;\theta_p^*,\theta_c^*)}$

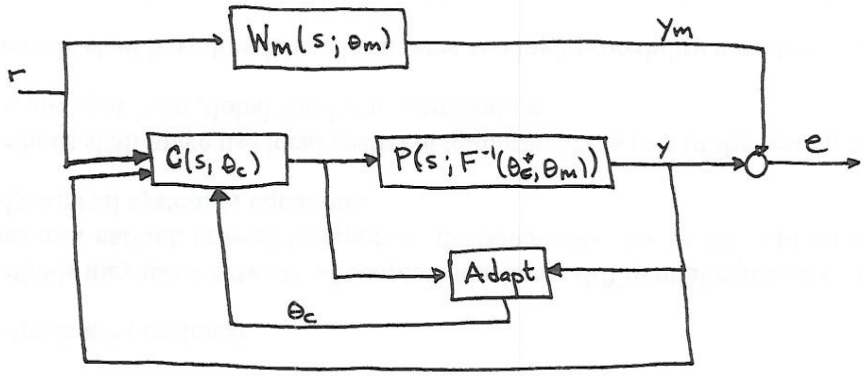
satisfies: $N_m(s;\theta_m) = N(s;\theta_p^*,\theta_c^*)$ & $D_m(s;\theta_m) = D(s;\theta_p^*,\theta_c^*)$

$\Rightarrow$

find $\theta_c = F(\theta_p, \theta_m)$ to achieve $N_m(s;\theta_m) = N(s;\theta_p^*, F(\theta_p^*,\theta_m))$ and

$D_m(s;\theta_m) = D(s;\theta_p^*, F(\theta_p^*,\theta_m))$.

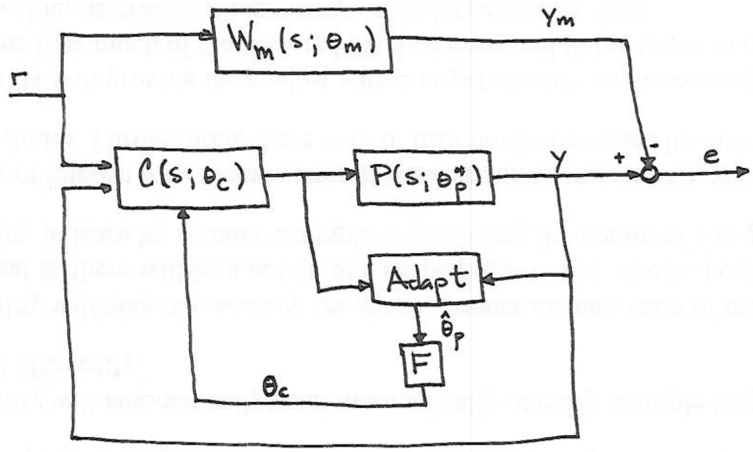
Direct Model Reference Adaptive Control:



recall that $\theta_c = F(\theta_p, \theta_m)$, so we define $\theta_p = F^{-1}(\theta_c, \theta_m)$.

goal is to estimate the controller parameters directly. ideally $\theta_c \to \theta_c^*$ as $t \to \infty$.

Indirect Model Reference Adaptive Control



come up with estimate $\hat{\theta}_p$ of θ_p, then choose controller parameters $\theta_c = F(\theta_p, \theta_m)$.

here, the goal is to have $\hat{\theta}_p \to \theta_p^*$ as $t \to \infty$.

Direct MRAC of First Order Systems

System/Plant : $\dot{x}(t) = a\,x(t) + b\,u(t)$ $\qquad x(0) = x_0$

unknown parameters, but sign(b) is known.

essential; think of the phase of the control.

Model : $\dot{x}_m(t) = a_m x_m(t) + b_m r(t)$ $\qquad x_m(0) = x_{m,0}$

$a_m < 0$

uniformly cts bdd input signal

GOAL: for uniformly bounded input $\{r(t) \in \mathbb{R} \mid |r(t)| \leq r_{max}\}$, define an adaptive feedback signal $u(t)$ such that the state $x(t)$ tracks $x_m(t)$ asymptotically with all signals bounded.

choose Controller: $u(t) = k_x(t)\,x(t) + k_r(t)\,r(t)$

$\Rightarrow$

$$\dot{x}(t) = (a + bk_x(t))\,x(t) + bk_r(t)\,r(t)$$

in order for controller to adapt to a solution, a unique solution must exist

$$a + bk_x^* = a_m \qquad \text{MATCHING CONDITIONS}$$
$$bk_r^* = b_m$$

- need unique solution to the matching conditions. we may not know it, but we need to know it's possible for adaptive control problem to be well-posed.

Define the error and error dynamics

$$e(t) = x(t) - x_m(t)$$

$\Rightarrow$

$$\dot{e}(t) = \dot{x}(t) - \dot{x}_m(t)$$

$$= (a + bk_x(t))x(t) + bk_r(t)r(t) - a_m x_m(t) - b_m r(t)$$

$$= (\underbrace{a + bk_x^*}_{a_m} + \underline{bk_x(t) - bk_x^*})x(t) - a_m x_m(t)$$

$$+ (\underline{bk_r(t) - bk_r^*} + \underbrace{bk_r^*}_{b_m})r(t) - b_m r(t)$$

CANCEL OUT

$$= a_m(x(t) - x_m(t)) + b(k_x(t) - k_x^*)x(t) + b(k_r(t) - k_r^*)r(t)$$

$$\dot{e}(t) = a_m e(t) + b\Delta k_x(t)x(t) + b\Delta k_r(t)r(t)$$

where $\Delta k_x(t) \equiv k_x(t) - k_x^*$

$\Delta k_r(t) \equiv k_r(t) - k_r^*$

Now, we are going to show asymptotic convergence with the candidate Lyapunov function,

$$V(e, \Delta k_x, \Delta k_r) = e^2(t) + |b|(\gamma_x^{-1}\Delta k_x^2(t) + \gamma_r^{-1}\Delta k_r^2(t))$$

$\Rightarrow$

$$\dot{V} = 2e(t)\dot{e}(t) + 2|b|\gamma_x^{-1}\Delta k_x(t)\Delta\dot{k}_x(t) + 2|b|\gamma_r^{-1}\Delta k_r(t)\Delta\dot{k}_r(t)$$

$$= 2e(t)[a_m e(t) + b\Delta k_x(t)x(t) + b\Delta k_r(t)r(t)]$$

$$+ 2|b|\gamma_x^{-1}\Delta k_x(t)\Delta\dot{k}_x(t) + 2|b|\gamma_r^{-1}\Delta k_r(t)\Delta\dot{k}_r(t)$$

$$= 2a_m e^2(t) + 2|b|\Delta k_x(t)[\gamma_x^{-1}\Delta\dot{k}_x(t) + \mathrm{sign}(b)e(t)x(t)]$$

$$+ 2|b|\Delta k_r(t)\underbrace{[\gamma_r^{-1}\Delta\dot{k}_r(t) + \mathrm{sign}(b)e(t)r(t)]}_{\text{best if these were to vanish.}}$$

unknown

$\Rightarrow$ to get desired outcome define

$$\Delta \dot{k}_x(t) = \dot{k}_x(t) \equiv -\,\mathrm{sign}(b)\,\gamma_x\, e(t)\, x(t)$$
$$\Delta \dot{k}_r(t) = \dot{k}_r(t) \equiv -\,\mathrm{sign}(b)\,\gamma_r\, e(t)\, r(t)$$

Adaptation Law.

$\Rightarrow$

$$\dot{V} = 2 a_m e^2(t)$$

$\Rightarrow$ $a_m < 0$

$$\dot{V} = -2|a_m| e^2(t) \leq 0$$

negative semi-definite.

$\Rightarrow$

Equilibrium is stable, the signals $e(t)$, $\Delta k_x(t)$, $\Delta k_r(t)$ are bounded

$\Rightarrow$

$x(t)$ is bounded since $e(t)$ and $x_m(t)$ are bounded.

Can we conclude asymptotic stability? Well, ...

$$\ddot{V} = -4|a_m|\, e(t)\, \dot{e}(t)$$

both bounded

$\Rightarrow$

$\ddot{V}$ bounded

$\Rightarrow$ Barbalat + Corollary

Tracking error is asy. stable, $e(t) \to 0$ as $t \to \infty$

(parameters are only gauranteed bounded)

Of note: ① needed a Lyapunov function whose second time derivative was bounded.

$\rightarrow$ adaptive control design is an inverse Lyapunov design methodology.

② sign(b) was needed. Essential to providing control in the proper direction.

Indirect MRAC of First Order Systems

- Idea is to estimate the plant parameters and not the controller parameters.
  Setup is going to differ.

Plant/System: $\dot{x}(t) = a x(t) + b u(t) \qquad x(0) = x_0$

unknown parameters,
but sign(b) is known,
and a conservative lower bound for $|b|$ is known.
* without loss of generality, assume $b > \bar{b} > 0$.

Model: $\dot{x}_m(t) = a_m x_m(t) + b_m r(t) \qquad x_m(0) = x_{m_0}$

$a_m < 0$

We are going to try to derive the ~~ad~~ controller by massaging $\hat{a}(t)$ & $\hat{b}(t)$ into $\dot{x}(t)$ & then find $\Delta a(t)/\Delta b(t)$ in $\dot{x}(t)$.

$$\dot{x}(t) = a x(t) + b u(t)$$

1. get estimate $\hat{b}(t)$ into equation

$$= a x(t) + \hat{b}(t) u(t) + (b - \hat{b}(t)) u(t)$$

$$= a x(t) + \hat{b}(t) u(t) - \Delta b(t) u(t)$$

$\Delta b(t) \equiv \hat{b}(t) - b$

$$= a x(t) + \underbrace{(a_m x(t) + b_m r(t) - \hat{a}(t) x(t))}_{\text{to get this, need for}} - \Delta b(t) u(t)$$

2. get ref. dynamics & estimate $\hat{a}(t)$ into equation through control.

$$u(t) \equiv \frac{1}{\hat{b}(t)} \left[a_m x(t) + b_m r(t) - \hat{a}(t) x(t) \right]$$

$$= a_m x(t) + b_m r(t) - \Delta a(t) x(t) - \Delta b(t) u(t)$$

$\Delta a(t) \equiv \hat{a}(t) - a$

3. get dynamics. $\Delta a = \Delta b = 0$ gives perfect tracking.

$\Rightarrow$ our controller should be

Controller: $u(t) = \frac{1}{\hat{b}(t)} \left[a_m x(t) + b_m r(t) - \hat{a} x(t) \right]$

- what's going on with $\dot{\hat{b}}(t)$?

for $\gamma_b u(t) e(t) \geq 0$ there are no problems since this increases $\hat{b}(t)$ (away from zero), or does nothing.

for $\gamma_b u(t) e(t) \leq 0$, get decrease of $\hat{b}(t)$.

Shouldn't go too far below lower bound.

The term $\dfrac{\bar{b} - \hat{b}(t)}{\hat{b}(t) - \bar{b} + \epsilon}$ starts to grow as $\hat{b}(t)$ goes below $\bar{b}$ and drives it back up, or at least takes it to the ~~term~~ solution to $\dot{\hat{b}}(t) = 0$.

It cannot drive $\hat{b}(t)$ below $(\bar{b} - \epsilon)$ since that is where the denominator blows up.

$\Rightarrow$

$\hat{b}(t)$ is bounded away from zero $\hat{b}(t) > \bar{b} - \epsilon > 0$.

$\Rightarrow$

feedback is OK and all affected differential equations are Lipschitz cts $\Rightarrow$ uniqueness is gavranteed.

also means that $\dot{\hat{b}}(t)$ is cts on domain of solution $\hat{b}(t)$.

$\Rightarrow$

May continue with next step: Lyapunov analysis!

* note the division by $\hat{b}(t)$ in the controller. We are going to have to ensure that $\hat{b}(t)=0$ never happens. Let's continue keeping this in mind.

OK, moving on let's define the error and error dynamics
$\Rightarrow$

$$e = x - x_m$$

$$\dot{e} = \dot{x} - \dot{x}_m$$

substitution

$$\dot{e}(t) = a_m e(t) - \Delta a(t) x(t) - \Delta b(t) u(t)$$

Inspired by direct MRAC, consider

$$\Delta \dot{a}(t) = \dot{\hat{a}}(t) \equiv \gamma_a x(t) e(t)$$

We can't quite do the same because of the constraint that $\hat{b}(t)=0$ can never happen. We need to keep adaptation dynamics away from zero. The following signal should do the trick:

$$\Delta \dot{b}(t) = \dot{\hat{b}}(t) \equiv \begin{cases} \gamma_b u(t) e(t) & \text{if } \hat{b}(t) \geq \bar{b} \\ \gamma_b u(t) e(t) + \dfrac{\bar{b} - \hat{b}(t)}{\hat{b}(t) - \bar{b} + \epsilon} & \text{if } \hat{b}(t) < \bar{b} \end{cases}$$

Adaptation

γ_a, γ_b adaptation gains

$\epsilon > 0$ tolerance. Should be small enough so that $\bar{b} - \epsilon > 0$

Define $V(e(t), \Delta a(t), \Delta b(t)) = e^2(t) + \gamma_a^{-1}\Delta a^2(t) + \gamma_b^{-1}\Delta b^2(t)$

$\Rightarrow$

$$\dot{V} = 2e(t)\dot{e}(t) + 2\gamma_a^{-1}\Delta a(t)\Delta\dot{a}(t) + 2\gamma_b^{-1}\Delta b(t)\Delta\dot{b}(t)$$

$$= 2e(t)\left[a_m e(t) - \Delta a(t)x(t) - \Delta b(t)u(t)\right] + 2\cancel{\gamma_a^{-1}}\Delta a(t)\cdot\cancel{\gamma_a}x(t)e(t) + 2\gamma_b^{-1}b(t)\dot{\Delta b}(t)$$

$$= 2a_m e^2(t) + 2\Delta a(t)\left(e(t)x(t) - \cancel{e(t)x(t)}\right) + 2\Delta b(t)\left(\gamma_b^{-1}\dot{\Delta b}(t) - u(t)e(t)\right)$$

$$\dot{V} = 2a_m e^2(t) + 2\Delta b(t)\left(\gamma_b^{-1}\dot{\Delta b}(t) - u(t)e(t)\right)$$

$\Rightarrow$

$$\dot{V} = \begin{cases} 2a_m e^2(t) & \text{if } \hat{b}(t) \geq \bar{b} \\ \underbrace{2a_m e^2(t)}_{<0} + \underbrace{\underbrace{2\Delta b(t)}_{<0}\,\underbrace{\frac{\bar{b}-\hat{b}(t)}{\hat{b}(t)-\bar{b}+\epsilon}}_{>0}}_{<0} & \text{if } \hat{b}(t) < \bar{b} \end{cases}$$

$\Rightarrow$

$\dot{V} \leq 0$ negative semi-definite.

$\Rightarrow$

Lyapunov stable

$\Rightarrow$

all terms bounded & $\ddot{V}$ bounded.

$\Rightarrow$

$e \to 0$ as $t \to \infty$

* results are not global over $(x, \hat{a}, \hat{b})$

have to worry about this.

- choice of ϵ kind of important.

More First Order, Scalar Direct MRAC:

Plant/System: $\dot{x} = ax + b(u + f(x))$

unknowns

sign(b) is known.

suppose that f(x) can be linearly parametrized in terms of N bounded basis functions

$$f(x) = \sum_{1}^{N} \alpha_i \varphi_i(x) = \alpha \cdot \Phi(x) = \alpha^T \Phi(x)$$

regressor vector

Reference Model: $\dot{x}_m(t) = a_m x_m(t) + b_m r(t)$

Controller: $u(t) = k_x(t)\, x(t) + k_r(t)\, r(t) - \hat{\alpha}^T(t)\, \Phi(x(t))$

substitution leads to the same matching conditions as before:

$$a + bk_x^* = a_m$$
$$bk_r^* = b_m$$

* matching conditions not needed for α since complete system of equations exist for the α_i.

$\Rightarrow$

$$\dot{V} = 2e(t)\Big[a_m e(t) + b\big[\Delta k_x(t)x(t) + \Delta k_r(t)r(t) - \Delta\alpha^T(t)\Phi(x(t))\big]\Big]$$
$$+ 2|b|\left[\gamma_x^{-1}\Delta k_x(t)\Delta\dot{k}_x(t) + \gamma_r^{-1}\Delta k_r(t)\Delta\dot{k}_r(t) + \Delta\alpha^T(t)\Gamma_\alpha^{-1}\Delta\dot{\alpha}(t)\right]$$

$$= 2a_m e^2(t) + \underbrace{2|b|\Delta k_x(t)\left[e(t)x(t)\,\mathrm{sign}(b) + \gamma_x^{-1}\Delta\dot{k}_x(t)\right] + 2|b|\Delta k_r(t)\left[e(t)r(t)\,\mathrm{sign}(b) + \gamma_r^{-1}\Delta\dot{k}_r(t)\right] + 2|b|\Delta\alpha^T(t)\left[-e(t)\,\mathrm{sign}(b)\Phi(x(t)) + \Gamma_\alpha^{-1}\Delta\dot{\alpha}(t)\right]}_{\text{want these to vanish.}}$$

$\Rightarrow$

$$\left.\begin{aligned}
\Delta\dot{k}_x(t) &= \dot{k}_x(t) \equiv -\mathrm{sign}(b)\gamma_x e(t)x(t) \\
\Delta\dot{k}_r(t) &= \dot{k}_r(t) \equiv -\mathrm{sign}(b)\gamma_r e(t)r(t) \\
\Delta\dot{\alpha}(t) &= \dot{\hat{\alpha}}(t) \equiv \mathrm{sign}(b)e(t)\Gamma_\alpha\Phi(x(t))
\end{aligned}\right\} \text{Adaptation}$$

$\Rightarrow$

$$\dot{V} = -2|a_m|e^2(t) \leq 0 \quad \text{negative semi-definite.}$$

$\vdots$

and following the same logic as before, $e(t) \to 0$ as $t \to \infty$.

* radial unboundedness of V $\Rightarrow$ globally asymptotically stable to E.

$$E = \{(e, \Delta k_x, \Delta k_r, \Delta\alpha) \mid \dot{V} = 0\}$$

$\Rightarrow$

$$E = \{(e, \Delta k_x, \Delta k_r, \Delta\alpha) \mid e = 0\}$$

The same can be done w/ indirect MRAC:

Plant/System: $\dot{x}(t) = a x(t) + b\left(u(t) + f(x(t))\right)$ $\quad x(0) = x_0$

Reference Model: $\dot{x}_m(t) = a_m x_m(t) + b_m r(t)$ $\quad x_m(0) = x_{m,0}$

Controller: $u(t) = \frac{1}{\hat{b}(t)}\left[a_m x(t) + b_m r(t) - \hat{a}(t)x(t) - \hat{\alpha}^T(t)\Phi(x(t))\right]$

Adaptation:

$$\dot{\hat{a}}(t) = \gamma_a e(t) x(t)$$

$$\dot{\hat{b}}(t) = \begin{cases} \gamma_a e(t) u(t) & \text{if } \hat{b}(t) \geq \bar{b} \\ \gamma_a e(t) u(t) + \dfrac{\bar{b} - \hat{b}(t)}{\hat{b}(t) - \bar{b} + \epsilon} & \text{if } \hat{b}(t) < \bar{b} \end{cases}$$

$$\dot{\hat{\alpha}}(t) = \Gamma_\alpha \Phi(x(t)) e(t)$$

* other update methods can be used for $\dot{\hat{b}}(t)$ so long as there is an appropriate Lyapunov function & one can gaurantee no zero crossings.