# Stability Theory for NonAutonomous Systems

We've already seen how time-varying components can fundamentally alter a dynamical system. Recall,

$$\dot{x} = -x \quad \text{vs.} \quad \dot{x} = -x + \delta \sin(t)$$

$\Downarrow$ exp. stable $\qquad\qquad$ $\Downarrow$ bounded

There are plenty of examples,

1] $\quad \dot{x} = \begin{bmatrix} -1 & g(t) \\ 0 & -1 \end{bmatrix} x \qquad x(0) = x_0$

$\hookrightarrow$ eigenvalues are both $-1$.

notice that $x_2$ is decoupled

$$\Rightarrow \quad x_2(t) = x_2(0)\, e^{-t}$$

$$\Rightarrow \quad \dot{x}_1(t) = -x_1(t) + g(t)\, x_2(0)\, e^{-t}$$

if $\quad g(t) = t \qquad$ all is good.

$g(t) = e^{t} \qquad$ all is OK (fixed point moves)

$g(t) = e^{2t} \qquad$ system goes unstable, escapes to $\infty$ as $t \to \infty$.

2] can affect convergence / stability type

$$\dot{x} = -x \qquad \text{vs.} \qquad \dot{x} = -\frac{x}{1+t}$$

$$\Downarrow \qquad\qquad \Downarrow$$

$$x(t) = x_0 e^{-t} \qquad\qquad x(t) = \frac{1+t_0}{1+t} x_0$$

$$\Downarrow \qquad\qquad \Downarrow$$

exp. stable. asy. stable

(not uniformly asy. stable)

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To extend Lyapunov theory to the time-varying case, gonna have to make some appropriate definitions and assumptions. First off, our system is

$$\dot{x} = f(x,t) \qquad x(t_0) = x_0 \,, \quad x \in D \tag{$\dagger$}$$

where $f : D \times [t_0, \infty)$ is piecewise continuous in $t$ and Lipschitz cts in $x$ on $D \times [t_0, \infty)$ uniformly in $t$, for $D \subset \mathbb{R}^n$ containing the origin.

Definition. A scalar time-varying function $V(x,t)$ is <u>(locally) positive definite</u> if $V(0,t) = 0$ and there exists a time-invariant, scalar, positive definite function $W_1(x)$ such that (local to the origin)

$$\forall\, t \geq t_0 \,, \quad V(x,t) \geq W_1(x) .$$

Definition. A scalar time-varying function $V(x,t)$ is <u>decrescent</u> if $V(0,t) = 0$ and there exists a time-invariant, scalar, positive definite function $W_2(x)$ such that (local to the origin)

$$\forall\, t \geq t_0 \,, \quad W_2(x) \geq V(x,t)$$

- both can be extended to consider global versions and/or semi-definiteness.

- $W_1$ & $W_2$ will be used to sandwhich $V$ for assisting in stability analysis.

We will also make use of the following concepts:

**Definition.** A continuous function $\varphi : [0, R) \to \mathbb{R}^+$ is said to belong to <u>class K</u> if

(i) $\varphi(0) = 0$ , and

(ii) $\varphi$ is strictly increasing on $[0, R)$

It is of <u>class $K_\infty$</u> if $R = \infty$ and $\varphi(r) \to \infty$ as $r \to \infty$.

* Ioannou & Sun use $KR$ for $K_\infty$.

- also exists notation $K_R$ and $K[0,R)$. describes interval of definition.

**Definition.** A continuous function $\psi : [0, R) \times \mathbb{R}^+$ belongs to <u>class $KL$</u> if

(i) $\psi(r, \tau)$ is class $K$ w/respect to $r$ for each fixed $\tau$.

(ii) $\psi(r, \tau)$ is decreasing function of $\tau$ for each fixed $r$ with $\psi(r,\tau) \to 0$ as $\tau \to \infty$.

It is of <u>class $KL_\infty$</u> if it is $KL$ and $\psi(r,\tau)$ is $K_\infty$ w/respect to $r$.

Examples.

| | | |
|---|---|---|
| class $K$ | $\varphi(r) = \tan(r)$ | for $r \in [0, \pi/2)$ |
| class $K_\infty$ | $\varphi(r) = kr$ | for $r \in [0, \infty)$ |
| | $\varphi(r) = \log(1+r)$ | for $r \in [0, \infty)$ |
| class $KL_\infty$ | $\psi(r,\tau) = kre^{-\tau}$ | for $k > 0$. |

Properties of K & KL

$$\varphi \in K[0,R) \Rightarrow \varphi^{-1} \quad K[0,\varphi(R))$$

$$\varphi \in K_\infty \Rightarrow \varphi^{-1} \quad K_\infty$$

$$\varphi_1 \varphi_2 \in K[0,R) \Rightarrow \varphi_1 \circ \varphi_2 \in K$$

$$\varphi_1 \circ \varphi_2 \in K_\infty \Rightarrow \varphi_1 \varphi_2 \in K_\infty$$

$$\varphi \in K, \psi \in KL \Rightarrow \varphi \circ \psi \in KL$$

- take care of domains.

Theorem [ Lyapunov's Stability Theorem for NonAutonomous Systems ]

Consider the system (†) with an equilibrium at the origin.

Let $V \in C^1(D \times \mathbb{R}^+, \mathbb{R})$ be such that

$$W_2(x) \geq V(x,t) \geq W_1(x)$$

and

$$\frac{\partial V}{\partial t} + D_1 V(x,t) \cdot f(x,t) = 0 \qquad \forall t \geq 0,\ x \in D \subset \mathbb{R}^n$$

for $W_1$ and $W_2$ continuous and positive definite, and $D \subset \mathbb{R}^n$ containing the origin. Then $x=0$ is uniformly stable.

If it further holds that

$$\frac{\partial V}{\partial t} + D_1 V(x,t) \cdot f(x,t) \leq -W_3(x) \qquad \forall t \geq 0,\ x \in D$$

where $W_3$ is continuous and positive definite in $D$. Then $x=0$ is uniformly asymptotically stable. Moreover, letting $B_r \subset D$ and $c < \min_{\|x\|=r} W_1(x)$, every trajectory starting in $\Omega_c = \{ x \in B_r \mid W_2(x) \leq c \}$ satisfies

$$\|x(t)\| \leq \gamma(\|x(t_0)\|, t - t_0) \qquad \forall t \geq t_0 \geq 0$$

for some $\gamma \in KL$.

If $D = \mathbb{R}^n$ and $W_1(x)$ is radially unbounded, then $x=0$ is globally uniformly asymptotically stable.

Finally, if $\exists\, k_1, k_2, k_3 > 0$ and $p \geq 1$ such that

$$k_2 \|x\|_p^p \geq V(x,t) \geq k_1 \|x\|_p^p$$

$$\frac{d}{dt} V(x,t) \leq -k_3 \|x\|_p^p \qquad \forall x \in D,\ t \geq 0$$

then the origin is (locally) exponentially stable.

If $D = \mathbb{R}^n$ can hold, then $x=0$ is globally exponentially stable.

Example. (Getting KL from K)

Given $\varphi \in K[0,R)$ Lipschitz continuous, it is possible to construct a class KL function on $[0,R) \times \mathbb{R}^+$ by solving the ODE

$$\dot{z}(t) = -\varphi(z(t)) \qquad z(0) = z_0$$

for $z_0 \in [0,R)$.

- case in point: $\varphi(r) = kr$

$$\Rightarrow \quad \dot{z} = -kz \quad \Rightarrow \quad z(t) = z_0 e^{-kt}$$

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How do these K & KL functions relate to V?

Lemma. Let $V: \mathbb{R}^n \to \mathbb{R}$ be continuous and positive definite in a ball $B_r \subset \mathbb{R}^n$ of the origin. Then, there exist $\varphi_1, \varphi_2 \in K[0,R)$ Lipschitz continuous such that

$$\varphi_2(\|x\|) \geq V(x) \geq \varphi_1(\|x\|) \qquad \forall x \in B_r$$

If $V$ is defined over all of $\mathbb{R}^n$ and is radially unbounded, then $\varphi_1, \varphi_2 \in K_\infty$ and $r$ can be anything, e.g., $r \in [0,\infty)$.

Example. A fairly straightforward example is

$$V(x) = x^T P x$$

we get

$$\lambda_{min}(P)\,\|x\|^2 \le x^T P x \le \lambda_{max}(P)\,\|x\|^2$$

$\Rightarrow$

$$\varphi_1(r) = \lambda_{min}(P) \cdot r$$

$$\varphi_2(r) = \lambda_{max}(P) \cdot r$$

The class K functions also relate to stability and can be useful in helping prove aspects of the Lyapunov Direct Method (in both cases).

# Stability and the K|KL functions.

Equilibrium point $x=0$ for (†) is

- *uniformly stable* if and only if $\exists\, \alpha \in K,\ c>0 : \|x(t)\| \le \alpha(\|x(t_0)\|) \quad \forall t \ge t_0 \ge 0$ and $\|x(t_0)\| < c$. $c$ is independent of $t_0$.

- *uniformly asymptotically stable* if and only if $\exists\, \psi \in KL[0,c)$, with $c$ independent of $t_0$, such that

$$\|x(t)\| \le \psi(\|x(t_0)\|, t-t_0) \quad \forall t \ge t_0 \ge 0 \ \&\ \|x(t_0)\| < c$$

- *globally asymptotically stable* if it is uniformly asymptotically stable and $\psi \in KL_\infty$.

  - exponential stability holds when the KL function is $\psi(r,\tau) = k r e^{-\lambda \tau}$.

Examples.

1] $\dot{x} = -(1 + \sin^2(t))\, x^3$

choose candidate Lyapunov function

$$V(x) = \frac{1}{2} x^2$$

$\Rightarrow$

$$\dot{V}(x) = x\dot{x} = -(1 + \sin^2(t))x^4 \leq x^4$$

$$W_1(x) = W_2(x) = \frac{1}{2}x^2 \quad , \quad W_3(x) = x^4$$

$\Rightarrow$ globally uniformly asymptotically stable.

2] $\ddot{x}(t) + c(t)\dot{x}(t) + k_0\, x(t) = 0 \qquad x(0) = x_0 \;,\; \dot{x}(0) = \dot{x}_0 \;,\; c(t) > 0$

$$\dot{x}_1(t) = x_2(t)$$

$$\dot{x}_2(t) = -c(t)\, x_2(t) - k_0\, x_1(t)$$

choose candidate Lyapunov function with free parameters $\alpha$ & $b(t)$:

$$V(x,t) = \frac{1}{2}(\alpha x_1 + x_2)^2 + \frac{1}{2} b(t) x_1^2(t)$$

$\Rightarrow$

$$\dot{V}(x,t) = (\alpha x_1 + x_2)(\alpha \dot{x}_1 + \dot{x}_2) + b(t) x_1 \dot{x}_1 + \frac{1}{2}\dot{b}(t) x_1^2$$

$$= (\alpha x_1 + x_2)\big((\alpha - c(t))x_2 - k_0 x_1\big) + b(t) x_1 x_2 + \frac{1}{2}\dot{b}(t) x_1^2$$

$$= \big(\alpha^2 - \alpha c(t) - k_0 + b(t)\big) x_1 x_2 + (\alpha - c(t)) x_2^2 + \left(\frac{1}{2}\dot{b}(t) - \alpha k_0\right) x_1^2(t)$$

first off choose $b(t)$ to satisfy $\quad b(t) = k_0 - \alpha^2 + \alpha c(t)$

leading to

$$\dot{V}(x,t) = (\alpha - c(t))\, x_2^2 + \left(\tfrac{1}{2}\dot{b}(t) - \alpha k_0\right) x_1^2(t)$$

$$\Rightarrow \quad b(t) = \alpha c(t)$$

$$\dot{V}(x,t) = (\alpha - c(t))\, x_2^2 + \left(\tfrac{1}{2}\alpha \dot{c}(t) - \alpha k_0\right) x_1^2(t)$$

for negative definiteness, need the following to be possible:

$$c(t) > \alpha \quad \text{for some } \alpha.$$

$$\alpha k_0 > \tfrac{1}{2}\alpha \dot{c}(t)$$

$$\Rightarrow \quad c(t) > \alpha \quad \text{and} \quad \dot{c}(t) < 2k_0$$

so, if $c(t)$ has lower bound greater than zero and $\dot{c}(t)$ has upper bound less than $2k_0$, then the Lyapunov function

$$V(x,t) = \tfrac{1}{2}(\alpha x_1 + x_2)^2 + \tfrac{1}{2} b(t) x_1^2(t)$$

with

$$0 < \alpha < c(t) \quad \text{and} \quad \dot{c}(t) \le \beta < 2k_0$$

$$b(t) = k_0 - \alpha^2 + \alpha c(t)$$

can be used.

$\Rightarrow$ $\dot{V}(x,t) \le 0$ negative definite.

$\Rightarrow$ $c(t)$ upper bounded.

asymptotically stable.

why upper bound $c(t)$?

well, recall I need

$$W_1(x) \le V(x,t) \le W_2(x)$$

$$W_1(x) = \tfrac{1}{2}(\alpha x_1 + x_2)^2 + \tfrac{1}{2}\min_t b(t)\, x_1^2$$

$$W_2(x) = \tfrac{1}{2}(\alpha x_1 + x_2)^2 + \tfrac{1}{2}\max_t b(t)\, x_1^2$$

$c(t)$ not bdd $\Rightarrow$ $b(t)$ not bdd

$\Rightarrow$ $W_2$ cannot exist.

if $\exists t^* : \dot{c}(t^*) = 2k_0$, then can only show stability.

if $c(t) \to 0$ as $t \to \infty$, then can only show stability

(or, for any increasing sequence of time).

3] $\dot{x} = A(t)x(t)$ $x(t_0) = x_0$

- What conditions on A guarantee stability?

- the Ioannou & Sun book has an example where all $Re(\lambda(t)) < 0 \quad \forall t \geq 0$ is not enough to guarantee stability (Example 3.4.10, pg. 123).

This may be a bit conservative, but...

choose the candidate Lyapunov function

$$V(t) = x^T(t)\, x(t)$$

$\Rightarrow$

$$\dot{V}(t) = x^T(t) A^T(t) x(t) + x^T(t) A(t) x(t)$$

$$= x^T(t) \left( A^T(t) + A(t) \right) x(t)$$

for $\dot{V}(t)$ negative definite, need $\Delta = A^T(t) + A(t)$ negative def.

note the $\Delta$ is symmetric $(\Delta^T = \Delta) \Rightarrow$ all eigenvalues are real.

$$\dot{V}(t) = x^T(t) \Delta(t) x(t) < 0 \text{ definite}$$

$\Rightarrow$

all eigenvalues of $\Delta(t)$ need to satisfy $\lambda < 0$,

since

$$x^T(t) \Delta(t) x(t) \leq \lambda_{max} x^T(t) x(t)$$

$\Rightarrow$ if this does hold

$$\dot{V}(t) = x^T(t) \Delta(t) x(t) \leq \lambda_{max} x^T(t) x(t) = \lambda_{max} V(t) \leq 0$$

$\Rightarrow$

$$\|x(t)\|^2 = V(t) \leq V_0 e^{\lambda_{max} t}$$

$\Rightarrow$

$$\|x(t)\| \leq \sqrt{V_0}\, e^{\frac{\lambda_{max}}{2} t} \qquad (\text{recall } \lambda_{max} < 0)$$

What if we again get the case that $\dot{V}(x,t) \leq 0$ for a nonautonomous system in the sense that $\dot{V}$ is negative semi-definite?

- cannot conclude asymptotic stability
- cannot use LaSalle because of time dependence.

The following Lemma may help.

Lemma [Barbalat]. If the differentiable function $f(t)$ converges to a finite limit as $t \to \infty$ and $\dot{f}$ is uniformly continuous, then

$\dot{f} \to 0$ as $t \to \infty$.

* sometimes written as $\int^t f(\tau)d\tau$ and $f(t)$.

· Some "counter" examples:

1) $\dot{f} \to 0$ as $t \to \infty$ $\not\Rightarrow$ $f \to L$ as $t \to \infty$

$$\dot{f}(t) = \frac{\cos(\log(t))}{t} \to 0 \text{ as } t \to \infty \quad \text{but} \quad f(t) = \sin(\log(t)) \text{ has no limit as } t \to \infty.$$

2) $f \to L$ as $t \to \infty$ $\not\Rightarrow$ $\dot{f} \to 0$ as $t \to \infty$

$$f(t) = e^{-t}\sin^2(e^{2t}) \to 0 \text{ as } t \to \infty \quad \text{but} \quad \lim_{t\to\infty} \dot{f}(t) = \lim_{t\to\infty} 2e^{t}\sin(e^{2t}) - e^{-t}\sin(e^{2t}) \neq 0$$

* hinges on uniform continuity of $\dot{f}$.

Barbalat's Lemma is quite useful since it implies that:

Corollary. If a scalar function $V(x(t),t)$ satisfies the conditions

1) $V(x,t)$ is lower bounded

2) $\dot{V}(x,t)$ is negative semi-definite, and

3) $\dot{V}(x,t)$ is uniformly continuous in time,

then $V(x,t) \to 0$ as $t \to \infty$.

* means that trajectories converge to $E = \{ x \mid \dot{V}(x,t) = 0 \}$.

(as opposed to $M \subset E$, where $M$ is the largest invariant set in $E$)

* one way to show uniform continuity is to show that the function is differentiable and has bounded derivative. This is a sufficient condition.

so, $g \in C^1$ and $\dot{g}$ bounded $\Rightarrow$ $g$ uniformly cts.

for our case, this means

$V \in C^2(D \times \mathbb{R}^+ ; \mathbb{R})$ and $\ddot{V}$ bounded $\Rightarrow$ $\dot{V}$ uniformly cts.

Example.

Consider the following simple adaptive system,

$$x(t) = -x(t) + \theta(t) r(t)$$

$$\dot{\theta}(t) = -x(t) r(t)$$

and the following candidate Lyapunov function

$$V(x,\theta) = \frac{1}{2} x^2 + \frac{1}{2} \theta^2$$

$\Rightarrow$

$$V(x,\theta) = x\dot{x} + \theta\dot{\theta} = -x^2 - x\theta r(t) - x\theta r(t)$$

$$= -x^2 \leq 0$$

$\Rightarrow$ Lyapunov's Thm

nonautonomous system is stable (using $W_1 = W_2 = V$)

$\Rightarrow$

all signals starting off bounded remain bounded.

$\Rightarrow$

$$\ddot{V} = -2x\dot{x} = +2x^2 - x\theta r(t)$$

would like for this to be bounded.

$\Rightarrow$

only unknown quantity is $r(t)$.

if $r(t)$ bounded, then $\ddot{V}$ bounded $\Rightarrow$ $\dot{V}$ uniformly cts.

$\Rightarrow$ $r(t)$ bdd + Barbalat's Lemma

$$\dot{V} \to 0 \text{ as } t \to \infty$$

$\Rightarrow$

$$x \to 0 \text{ as } t \to \infty$$

$x(t)$ is asymptotically stable. (can only concluded boundedness of $\theta(t)$)